Variational Calculus and its Applications in Control Theory and Nano mechanics

Professor Sarthok Sircar
Department of Mathematics
Indraprastha Institute of Information Technology, Delhi

Lecture – 54

Conjugate Points / Jacobi Accessory Equations / Introduction to Optimal Control Theory

Part 6

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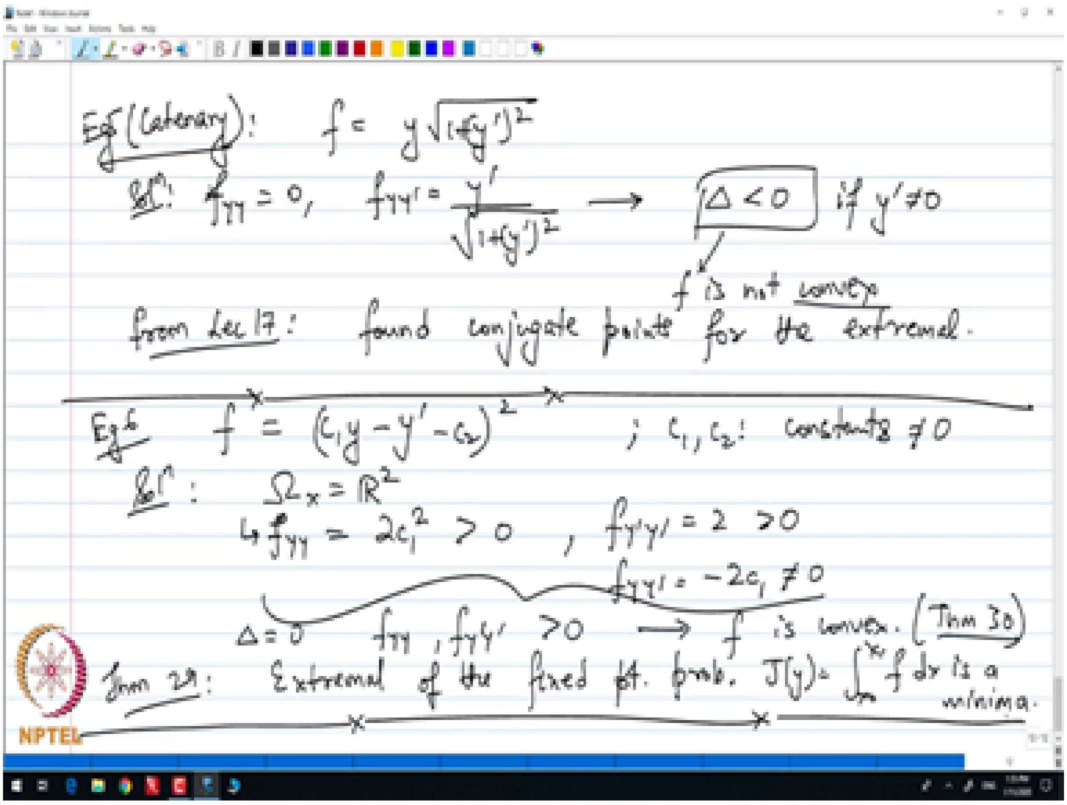

Now I am going to branch out and discuss an application of the calculus of variations namely, the application in Optimal Control Theory. We will see that in this example more often than not we are going to deal with functionals which has convex integrands. So, solving problems in optimal control theory will be a subset of whatever we have seen so far in terms of optimizing a functional.

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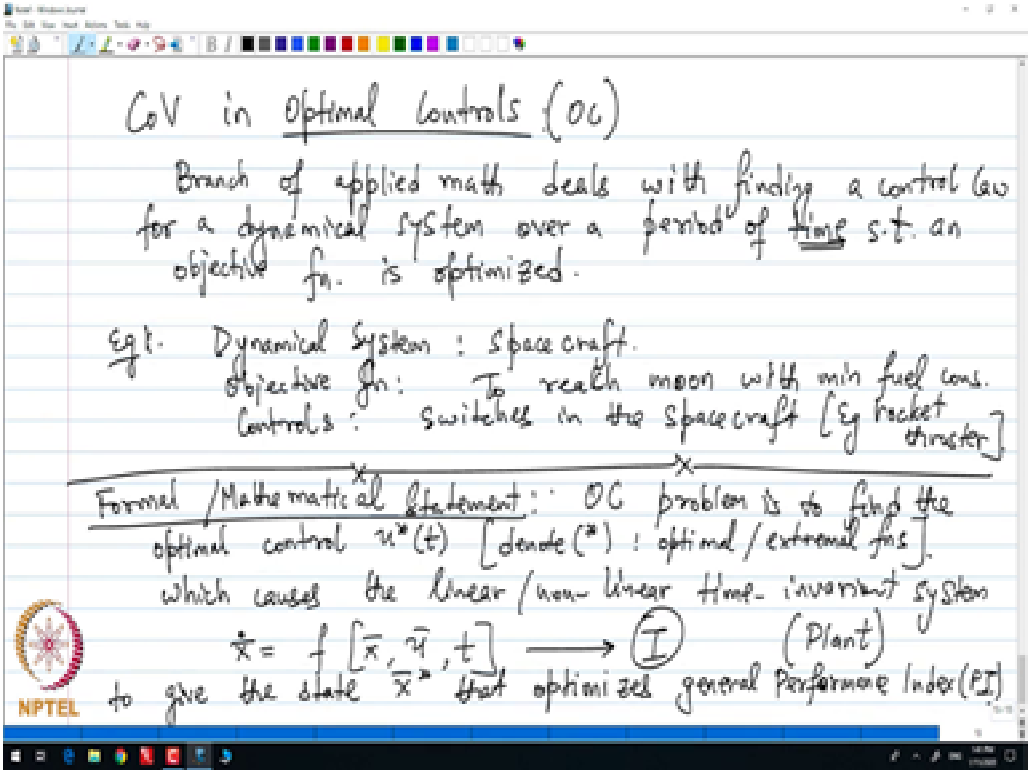

Calculus of variation in optimal controls (OC): So, what is this optimal control? It is a branch of applied mathematics with deals with finding a control law for a dynamical system over a period of time such that an objective function is optimized.

An example is as follows :The dynamical system could be spacecraft and objective function could be to reach moon from the planet earth with minimum fuel consumption and the controls are the switches in the spacecraft for example the rocket thruster.

Another dynamical system could be the nation itself, and let us say the objective function is to reduce unemployment and the controls could be the monetary and the fiscal policy which means the current topics of reducing unemployment in our country could possibly be tackled with our optimal control setup. So, let us now formalize our definition in the form of a mathematical statement. So, formal or mathematical statement: The OC problem is to find the optimal control $u^{\star}(t)$, (where $(^{\star})$ denote the optimal or extremal functions) which causes the linear or non-linear time invariant system.The system is denoted by the ODE

$$\dot{\bar{x}} = f\left[\bar{x}, \bar{u}, t\right] \qquad \textbf{(1)} \qquad \textbf{(Plant)}$$

to give the state $\bar{x}^{\star}$ x that optimizes the general Performance Index (PI).

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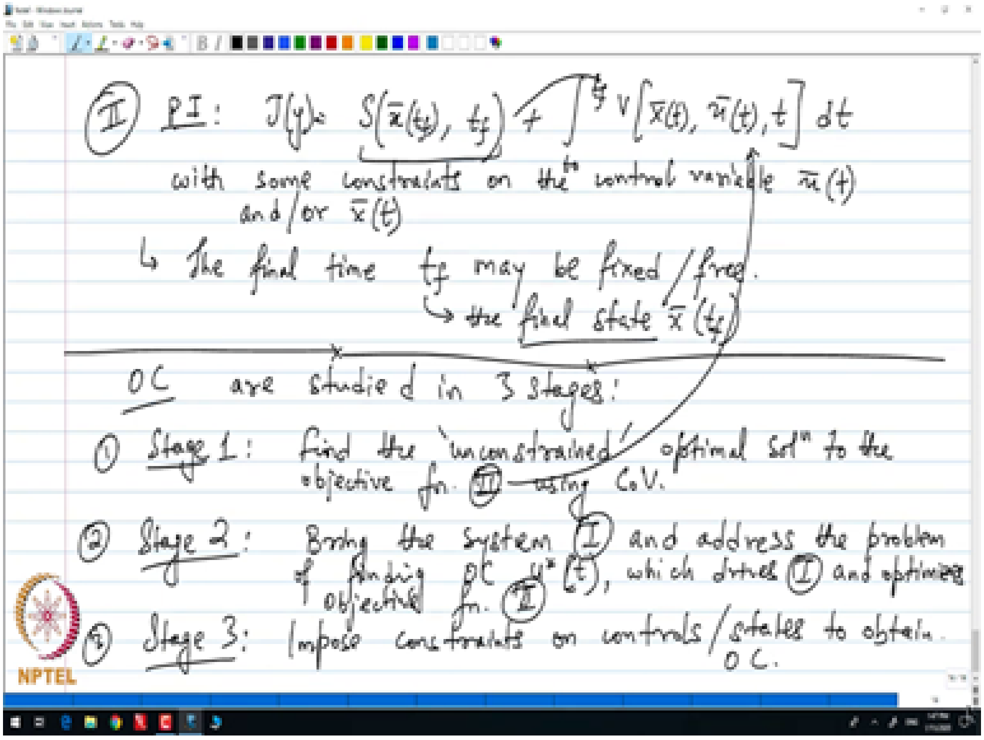

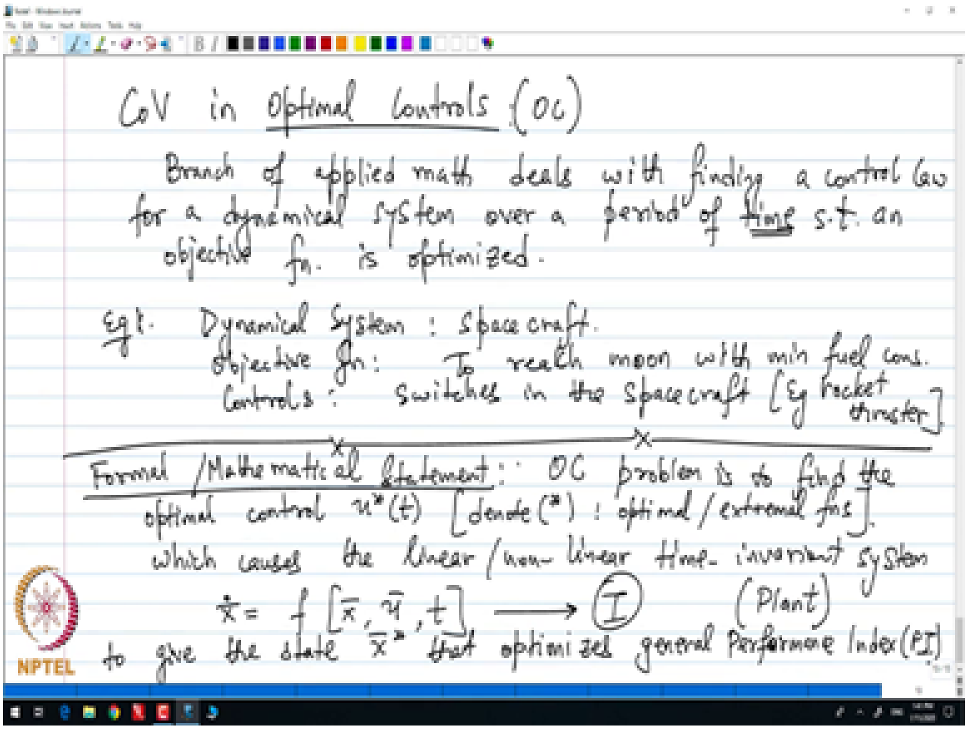

And, the Performance Index is given by the functional

$$J(y) = S(\bar{x}(t_f), t_f) + \int_{t_0}^{t_f} V[\bar{x}(t), \bar{u}(t), t]\, dt \tag{2}$$

Notice that in the mathematical terms, we are trying to optimizing the functional given by (2) subject to a non-holonomic constraint given by (1). So, we have to find these functions u given the constraints (1) so that the objective function (2) is optimized. So, this is our performance index with some constraints on the control variable $\bar{u}(t)$ and/or $\bar{x}(t)$ such that the final time t_f may be fixed or it is free. And the final state is denoted by $\bar{x}(t_f)$. So, we have basically introduced some physical terms so far in our calculus of variations notations here. Now , objective function is denoted by Performance Index, PI, and constraint is denoted by plant condition which is a non-holonomic constraint or the differential constraints. So, the OC problems are going to be studied in three stages. We will see that the step by step solution methodology follows very similar to the solution methodology of constraint optimization with non-holonomic constraints.

Stage 1 : We have to find the unconstrained optimal solution to the objective function (2) using calculus of variations.

Stage 2 : We are going to bring the system (1) and address the problem of finding the optimal control $u^\star(t)$ which drives (1) and optimizes the objective function (2).

Stage 3 : We impose constraints on controls and states to obtain optimal control solutions.

So, let us see what is this 3 stage solution methodology, but before that let me reduce the Performance Index in a form that we are familiar with.

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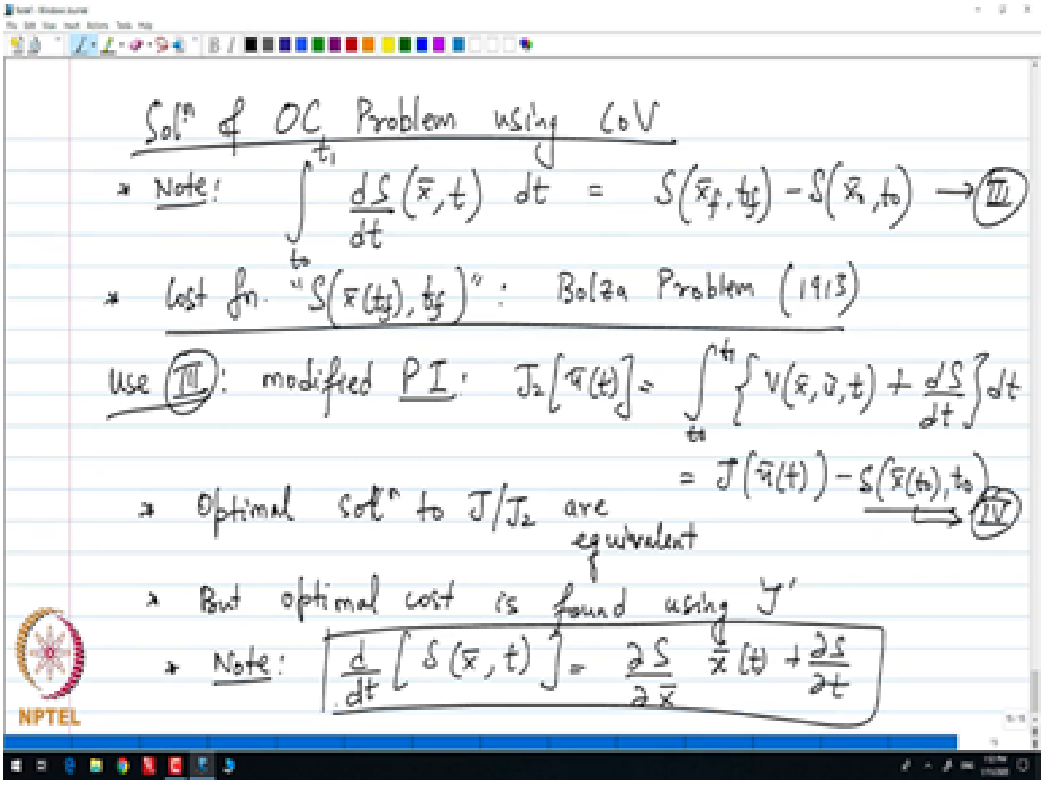

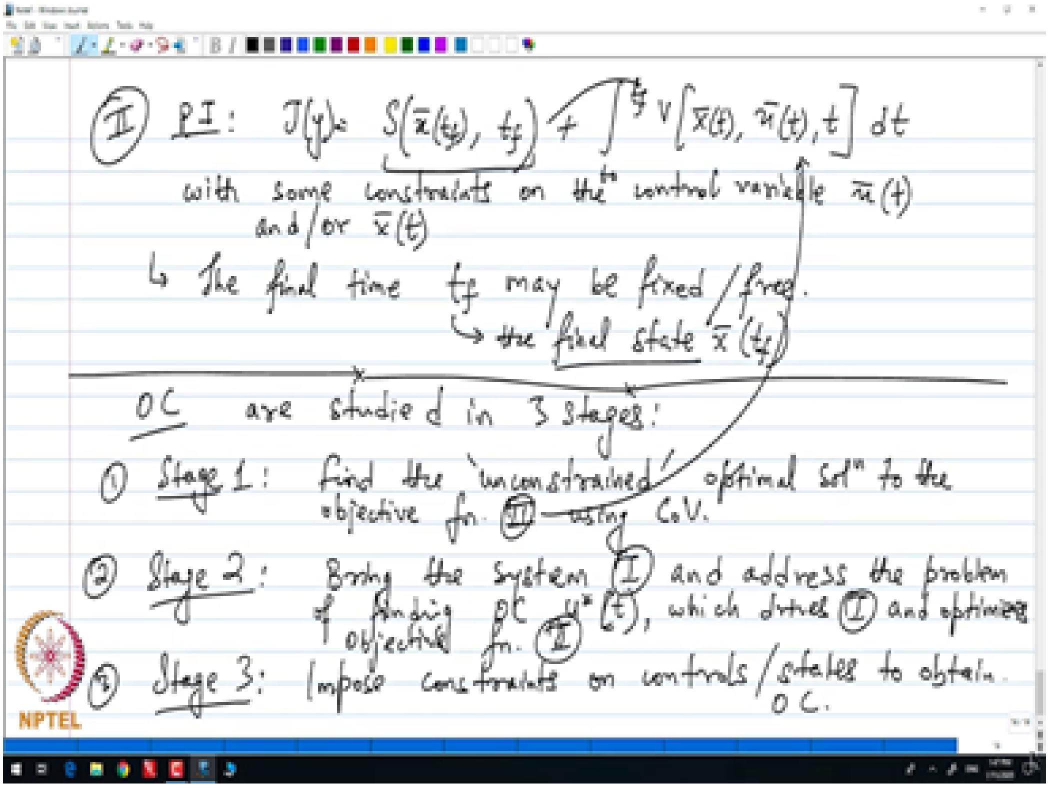

We are going to look at the solution of optimal control problems purely using variational techniques. Because this is a course on calculus of variations and not optimal controls. We note the following:

$$\int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}x} S(\bar{x}, t)\, dt = S(\bar{x}_f, t_f) - S(\bar{x}_0, t_0) \qquad \textbf{(3)}$$

In physical terms, the function S is known as the cost function. This objective function was introduced in 1913 by a Russian mathematician named Bolza. So, when we introduced this objective function including the cost function $S(\bar{x}(t_f), t_f)$ is also known as the Bolza problem.

Then use (3) and write down the modified performance index or the objective function

$$J_2[\bar{u}(t)] = \int_{t_0}^{t_1} \left\{ V(\bar{x}, \bar{u}, t) + \frac{\mathrm{d}S}{\mathrm{d}t} \right\} dt = J(\bar{u}(t)) - S(\bar{x}(t_0), t_0) \qquad \textbf{(4)}$$

Now we have absorbed the cost function inside the integral, but it turns out that this functional is slightly different from our original objective functional or the performance index.

Now, notice that the objective function J and J_2, they only vary by a constant which means that the optimal solution to J will be identically equal to the optimal solution to J_2. So, the optimal solution to J and J_2 are equivalent and hence the optimal control problem can either be found from optimizing J or J_2. But the optimal cost is found using J.

And also further note that $\frac{\mathrm{d}}{\mathrm{d}t}[S(\bar{x}, t)] = \frac{\partial S}{\partial \bar{x}} \dot{\bar{x}}(t) + \frac{\partial S}{\partial t}$,the dot represents the time derivative. Now I am going to highlight the 3-stage methodology of the solution to the optimal control problem in a step by step format.

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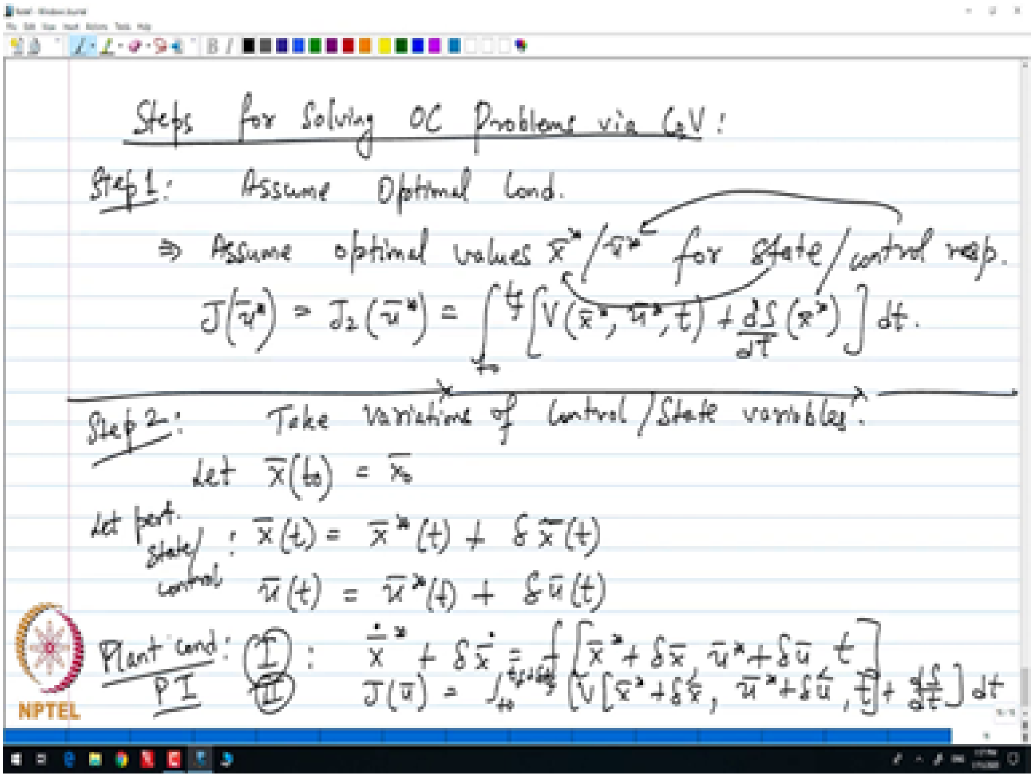

So, steps for solving optimal control problems via the calculus of variations. we will see that if we follow the step by step methodology, then almost all the problems in this topic are going to be solvable.

Step 1: We assume optimal conditions which means we assume the optimal values $\bar{x}^{\star}$ and $\bar{u}^{\star}$ for the state variables and for control variables respectively.

$$J\left(\bar{u}^{\star}\right) = J_2\left(\bar{u}^{\star}\right) = \int_{t_0}^{t_f} \left[V\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right) + \frac{\mathrm{d}S(\bar{x}^{\star})}{\mathrm{d}t}\right] dt$$

Step 2 : We take variations of the control and the state variables. Let us fix one point $\bar{x}\left(t_0\right) = \bar{x_0}$. So, let the perturbed state and control variables are as follows:

$$\bar{x}\left(t\right) = \bar{x}^{\star}\left(t\right) + \delta\bar{x}\left(t\right)$$

$$\bar{u}\left(t\right) = \bar{u}^{\star}\left(t\right) + \delta\bar{u}\left(t\right)$$

So,plant condition (1) reduces to the following for the perturbed variable:

$$\dot{\bar{x}}^{\star} + \delta\dot{\bar{x}} = f\left[\bar{x}^{\star} + \delta\bar{x}, \bar{u}^{\star} + \delta\bar{u}, t\right]$$

The objective function which is performance index is now written in the form:

$$J\left(\bar{u}\right) = \int_{t_0}^{t_f+\delta t_f} \left[V\left[\bar{x}^{\star} + \delta\bar{x}, \bar{u}^{\star} + \delta\bar{u}, t\right] + \frac{\mathrm{d}S}{\mathrm{d}t}\right] dt$$

And finally, we are going to solve this constraint problem by introducing the Lagrange multiplier function also known as the co-state vectors.

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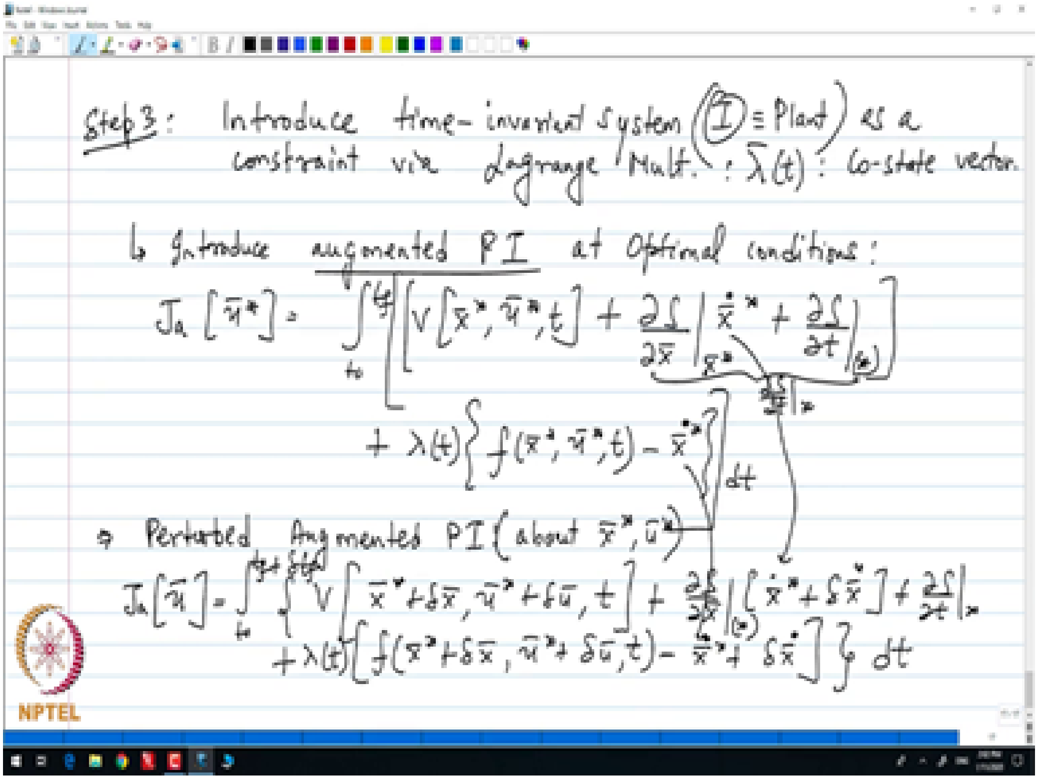

Step 3 : Introduce time invariant system (1) which is plant condition as a constraint via the Lagrange multiplier say $\bar{\lambda}(t)$ also known as the co-state vector. So, that is Lagrange multiplier function. Let me introduce the augmented performance index at optimal conditions are given by

$$J_a\left[\bar{u}^{\star}\right]=\int_{t_0}^{t_f}\left[\left[V\left[\bar{x}^{\star}, \bar{u}^{\star}, t\right]+\frac{\partial S}{\partial \bar{x}}\left.\right|_{\bar{x}^{\star}} \dot{\bar{x}}^{\star}+\frac{\partial S}{\partial t}\left.\right|_{\bar{x}^{\star}}\right]+\lambda(t)\left\{f\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right)-\dot{\bar{x}}^{\star}\right\}\right] d t$$

The perturbed augmented performance index about $(\bar{x}^{\star}, \bar{u}^{\star})$ can be written as follows:

$$J_a(\bar{u})=\int_{t_0}^{t_f+\delta t_f}\left[V\left[\bar{x}^{\star}+\delta \bar{x}, \bar{u}^{\star}+\delta \bar{u}, t\right]+\frac{\partial S}{\partial \bar{x}}\left.\right|_{\bar{x}^{\star}}\left[\dot{\bar{x}}^{\star}+\delta \dot{\bar{x}}\right]+\frac{\partial S}{\partial t}\left.\right|_{\bar{x}^{\star}}+\lambda(t)\left\{f\left(\bar{x}^{\star}+\delta \bar{x}, \bar{u}^{\star}+\delta \bar{u}, t\right)-\dot{\bar{x}}^{\star}+\delta \dot{\bar{x}}\right\}\right] d t$$

We have already seen the difference of the perturbed value with the original or the extremal value. And then from here, we will look at what is the first variation from this difference and we find the optimal control solution $\bar{u}^{\star}$ and the state variable $\bar{x}^{\star}$.