Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 56
Constrained Optimization in Optimal Control Theory Part 2

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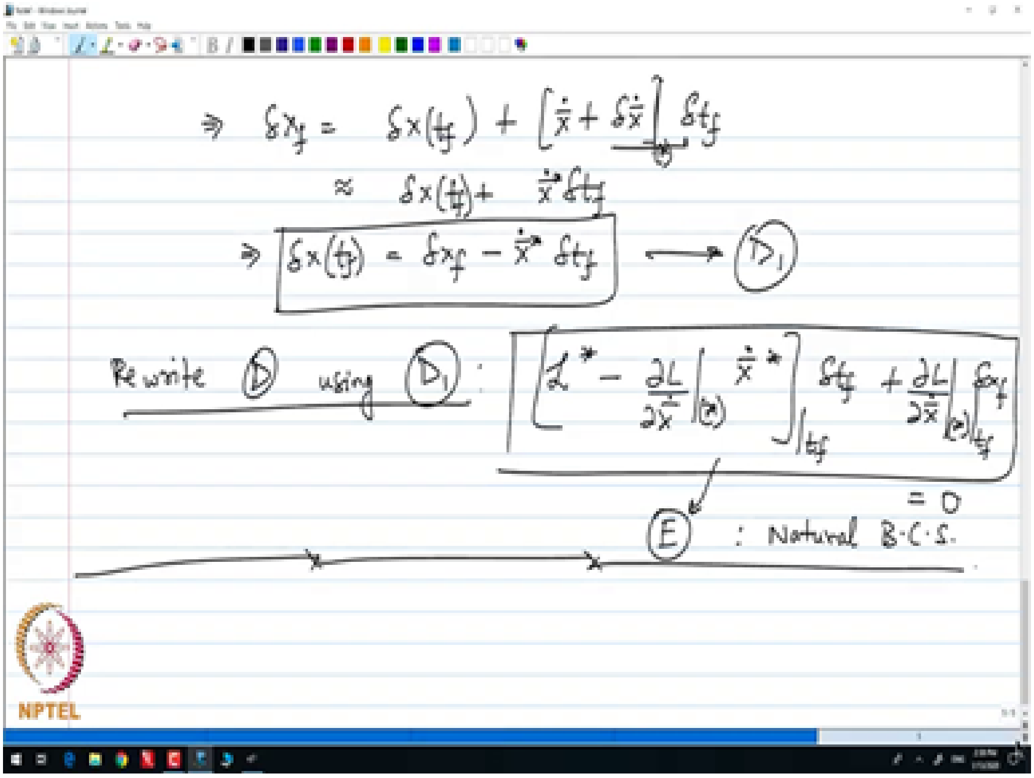

The final step is we use Hamiltonian formulation.

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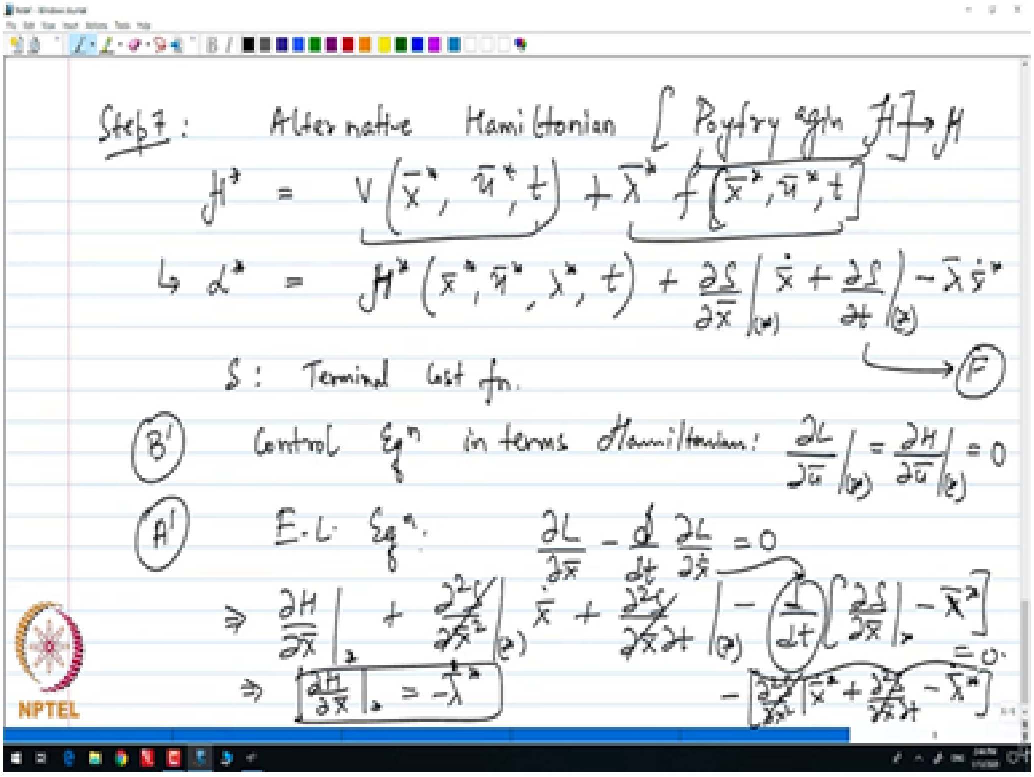

Step 7 : we use alternative Hamiltonian formulation or introduce the Pontryagin H function. So, the H evaluated at optimal condition is

$$H^{\star} = V\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right) + \bar{\lambda}^{\star} f\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right)$$

Then, we have to change Lagrangian to the Hamiltonian description. So, notice that Lagrangian can be written in the form of a Hamiltonian as foloows :

$$L^{\star} = H^{\star}\left(\bar{x}^{\star}, \bar{u}^{\star}, \lambda^{\star}, t\right) + \frac{\partial S}{\partial \bar{x}}\mid_{\bar{x}^{\star}} \dot{\bar{x}} + \frac{\partial S}{\partial t}\mid_{\bar{x}^{\star}} - \bar{\lambda}\dot{\bar{x}}^{\star} \qquad \textbf{(F)}$$

Then, recall that S is terminal cost function and then write down all constraints from the Lagrangian form to the Hamiltonian form. So, the control constraints in terms of Hamiltonian becomes

$$\frac{\partial L}{\partial \bar{u}}\mid_{\bar{x}^{\star}} = \frac{\partial H}{\partial \bar{u}}\mid_{\bar{x}^{\star}} = 0 \qquad \textbf{(B')}$$

we see that u only appears in H, so that change is quite simple.The Euler-Lagrange equation is changed as follows. Notice that the original E.L. equation in terms of the Lagrangian was the following:

$$\frac{\partial L}{\partial \bar{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right)\mid_{\bar{x}^{\star}} = 0$$

All we need to do is plug in L in the form of H and we get the following expression:

$$\frac{\partial H}{\partial \bar{x}}\mid_{\bar{x}^{\star}} + \frac{\partial^2 S}{\partial \bar{x}^2}\mid_{\bar{x}^{\star}} \dot{\bar{x}} + \frac{\partial^2 S}{\partial \bar{x}\partial t}\mid_{\bar{x}^{\star}} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial S}{\partial \bar{x}}\mid_{\bar{x}^{\star}} - \bar{\lambda}^{\star}\right] = 0 \qquad \textbf{(A')}$$

Now, this is after taking into account whatever variables are appearing. And then since circled quantity is the total time derivative, we use chain rule to make it a partial time derivative. So, the quantity $\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial S}{\partial \bar{x}}\mid_{\bar{x}^{\star}} - \bar{\lambda}^{\star}\right]$ reduces to $\frac{\partial^2 S}{\partial \bar{x}^2}\mid_{\bar{x}^{\star}} \dot{\bar{x}}^{\star} + \frac{\partial^2 S}{\partial \bar{x}\partial t} - \dot{\bar{\lambda}}^{\star}$

And after cancel out some similar term we get the following :

$$\frac{\partial H}{\partial \bar{x}}\mid_{\bar{x}^{\star}} = -\dot{\bar{\lambda}}^{\star}$$

Notice that Euler-Lagrange equation has reduced to an extremely simple form and finally, I complete the system by introducing co-state equation and the boundary condition.

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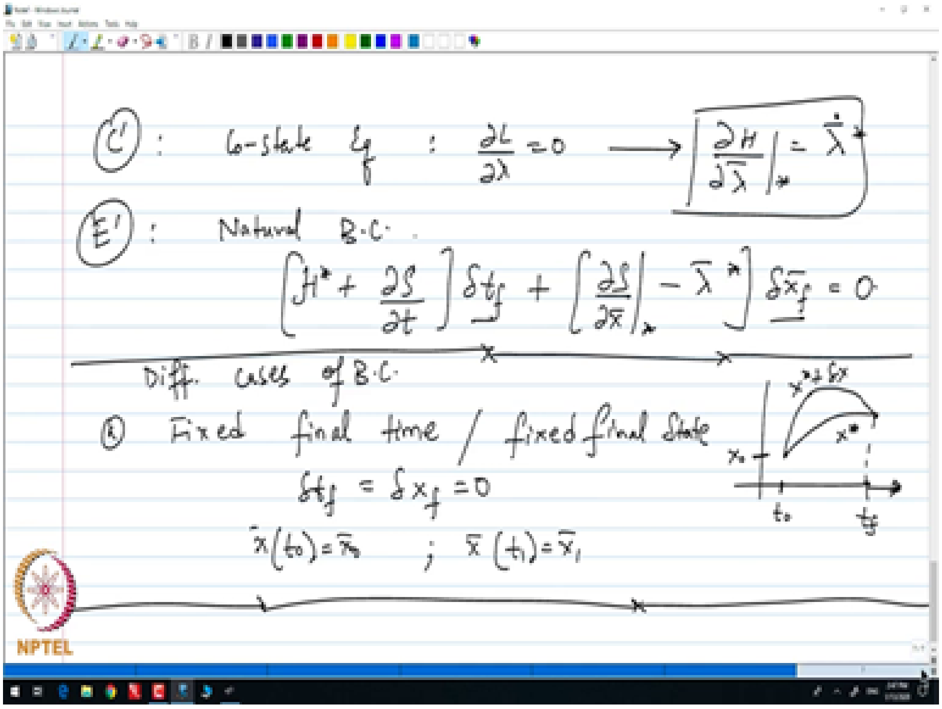

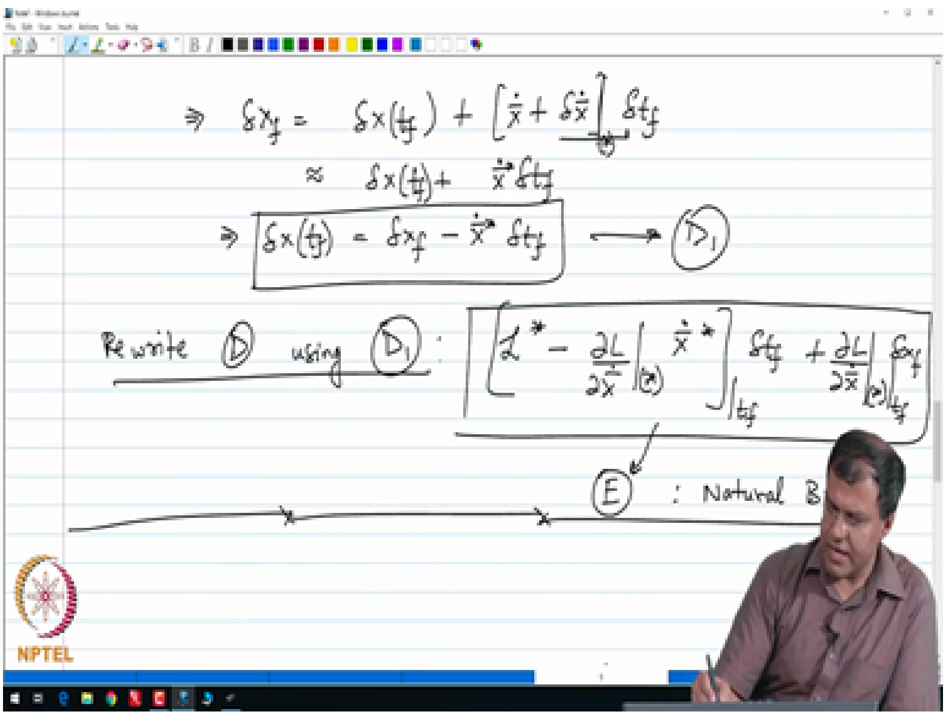

So, co-state equation $\frac{\partial L}{\partial \lambda} = 0$ is reduced to

$$\frac{\partial H}{\partial \bar{\lambda}} \mid_{\bar{x}^{\star}} = \dot{\bar{\lambda}}^{\star} \qquad \textbf{(C')}$$

Now the natural boundary condition reduces to the following form:

$$\left[H^{\star} + \frac{\partial S}{\partial t}\right] \delta t_f + \left[\frac{\partial S}{\partial \bar{x}} \mid_{\bar{x}^{\star}} - \bar{\lambda}^{\star}\right] \delta \bar{x}_f = 0 \qquad \textbf{(E')}$$

So, we have completed the description of the solution of this optimal control problem, we can again simplify problems into various sub-cases depending on whether our boundary points are fixed or variable. So, let me describe some of the simplified cases. So, different cases of boundary condition are as follows :

(a) Fixed time point and fixed state variables. Notice that, in our setup, we never change our initial reference point whether it is the optimal curve and whether it is the perturbed curve, it always starts with the same starting point. But let us say my first case is fixed final time so, t_f does not change and fixed final state which means that final point does not change.

And which means that natural boundary condition is trivially satisfied because $\delta t_f = 0$ and $\delta x_f = 0$ both variations are 0. The only conditions that we are going to get is the fixed point boundary conditions itself. So, $\bar{x}(t_0) = \bar{x_0}$ and $\bar{x}(t_1) = \bar{x_1}$. Then the next condition that we can describe is, it is fixed final time but variable final state which means that $t_f = 0$, the variation in time is 0, but the variation in the final state variable is not.

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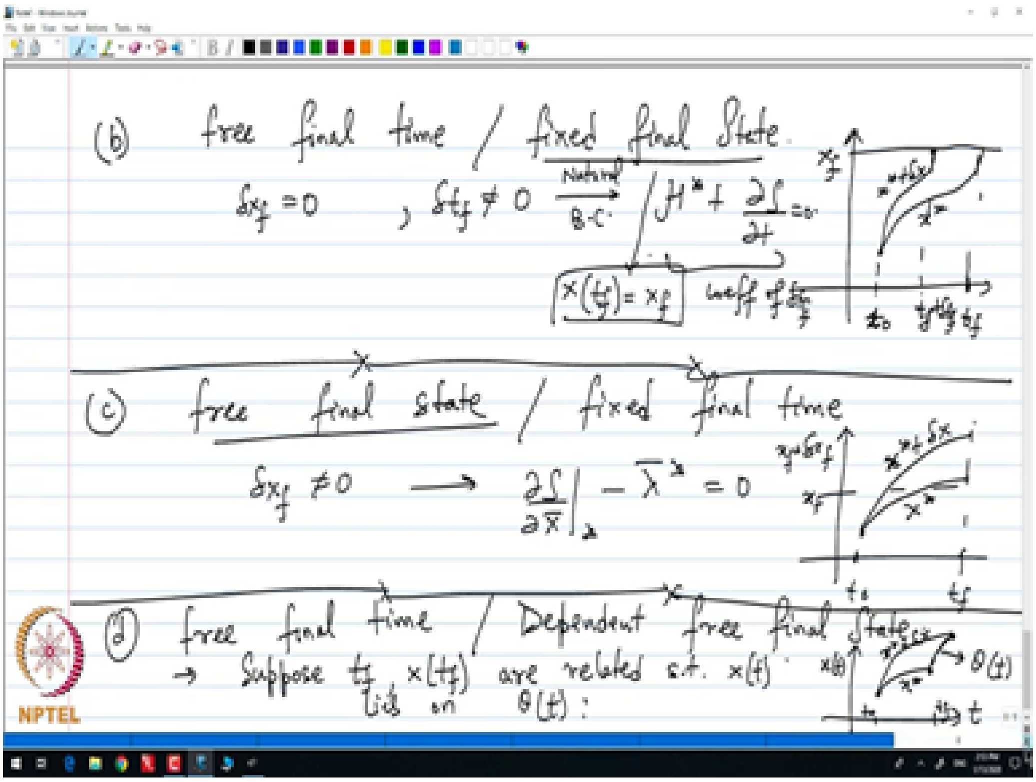

(b) Free final time but fixed final state. So, we have $\delta x_f = 0$; $\delta t_f \neq 0$. Let me just draw the diagram of this situation. So, let us say, this is time point t_0 and time point t_f and initial curve is let us say the following. This is $x^{\star}$, and $x^{\star}$ changes in such a way that final state does not change. But, let us say new perturbed quantity is $t_f + \delta t_f$. So, the final time changes but the final state does not change. In that case, $\delta t_f \neq 0$ which means our natural boundary condition kicks in and we get the following $H^\star + \frac{\partial S}{\partial t} = 0$ which is the coefficient of δt_f. Also we have the condition $x(t_f) = x_f$ that needs to be satisfy.

(c) Free final state and fixed final time the other way around . Let me draw the diagram in this situation again we fix t_0 and t_f and we get $x^\star$ and we have free final state, the other way around, but fixed final time and I see that this is the scenario.So, we have x_f and $x_f + \delta x_f$. In this scenario we have that $\delta x_f \neq 0$ and natural boundary condition kicks in and we get $\frac{\partial S}{\partial \bar{x}} |_{\bar{x}^\star} - \bar{\lambda}^\star = 0$.

(d) Free final time but dependent free final state and we see that this is also going to be $x(t)$ and t here from t_0 to t_f but now my curve moves along in such a way that the perturbed curves moves along in such a way that it follows a curve. So, let us say this is $x^\star + \delta x$ and perturbed curve is $\theta(t)$, where final point on the optimal curve and the perturbed curve lies. Suppose, final time point t_f and the state variable $x(t_f)$ are related such that $x(t)$ lies on $\theta(t)$.

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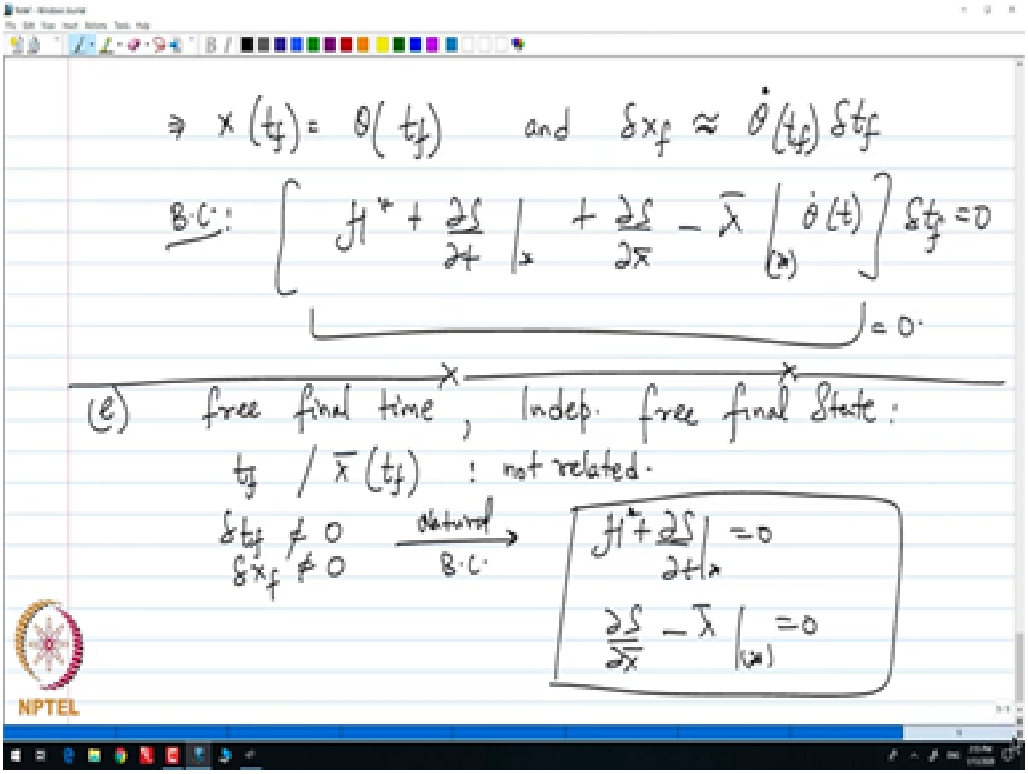

We see that $x(t_f) = \theta(t_f)$ and $\delta x_f \approx \dot{\theta}(t_f)\,\delta t_f$ where the $(\dot{})$ represents a derivative with respect to t . Then natural boundary condition will kick in and we get

$$\left[H^\star + \frac{\partial S}{\partial t} |_{\bar{x}^\star} + \frac{\partial S}{\partial \bar{x}} - \bar{\lambda} |_{\bar{x}^\star} \dot{\theta}(t)\right] \delta t_f = 0$$

(e) Free final time and independent free final state. If t_f and $\bar{x}(t_f)$ are not related which means neither $\delta t_f \neq 0$ nor $\delta x_f \neq 0$ .So, natural boundary condition will give us

$$H^\star + \frac{\partial S}{\partial t} |_{\bar{x}^\star} = 0$$

$$\frac{\partial S}{\partial \bar{x}} - \bar{\lambda} |_{\bar{x}^\star} = 0$$

These are all the steps that I had to mention, let us summarize all these steps, especially the steps via the Hamiltonian formulation, which is the easier out of the two methods.

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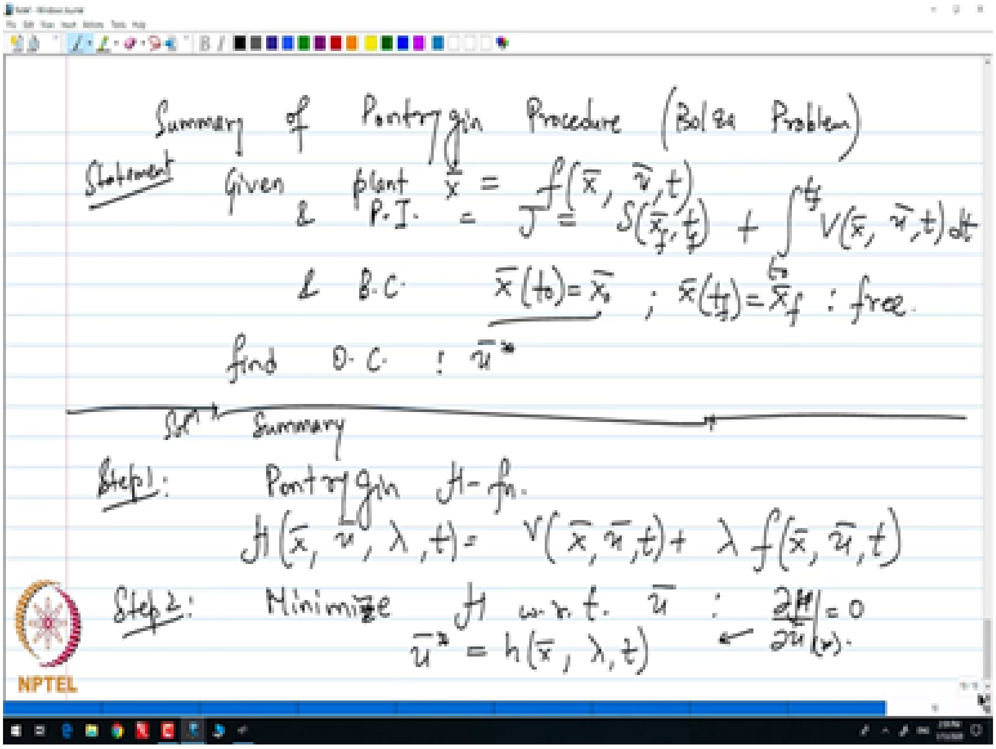

Summary of the Pontryagin procedure or the Bolza problem :

Statement : Given

Plant condition $\dot{\bar{x}} = f\left(\bar{x}, \bar{u}, t\right)$

and performance index $J = S\left(\bar{x_f}, t_f\right) + \int_{t_0}^{t_f} V\left(\bar{x}, \bar{u}, t\right) dt$

and the boundary conditions $\bar{x}\left(t_0\right) = \bar{x_0}$; $\bar{x}\left(t_f\right) = \bar{x_f}$ which may or may not be free. Generally, we fix the initial boundary condition. We have to find the optimal control which is given by $\bar{u}^{\star}$. The solution summary is as follows :

Step 1: We find the Pontryagin H - function.

$$H\left(\bar{x}, \bar{u}, \lambda, t\right) = V\left(\bar{x}, \bar{u}, t\right) + \lambda f\left(\bar{x}, \bar{u}, t\right) \text{ , that is Hamiltonian.}$$

Step 2 : we are going to minimize the Hamiltonian with respect to the control. Notice that we are writing the word minimize although we are taking the first derivative and the reason being we are dealing with all about convex functions, we have seen that when functions are convex our extremum is going to give us minimum. So, minimize H with respect to $\bar{u}$ by setting $\frac{\partial H}{\partial \bar{u}}\left.\right|_{\bar{x}^{\star}} = 0$. From this condition, I am going to get the optimal value of u which is $\bar{u}^{\star} = h\left(\bar{x}, \lambda, t\right)$

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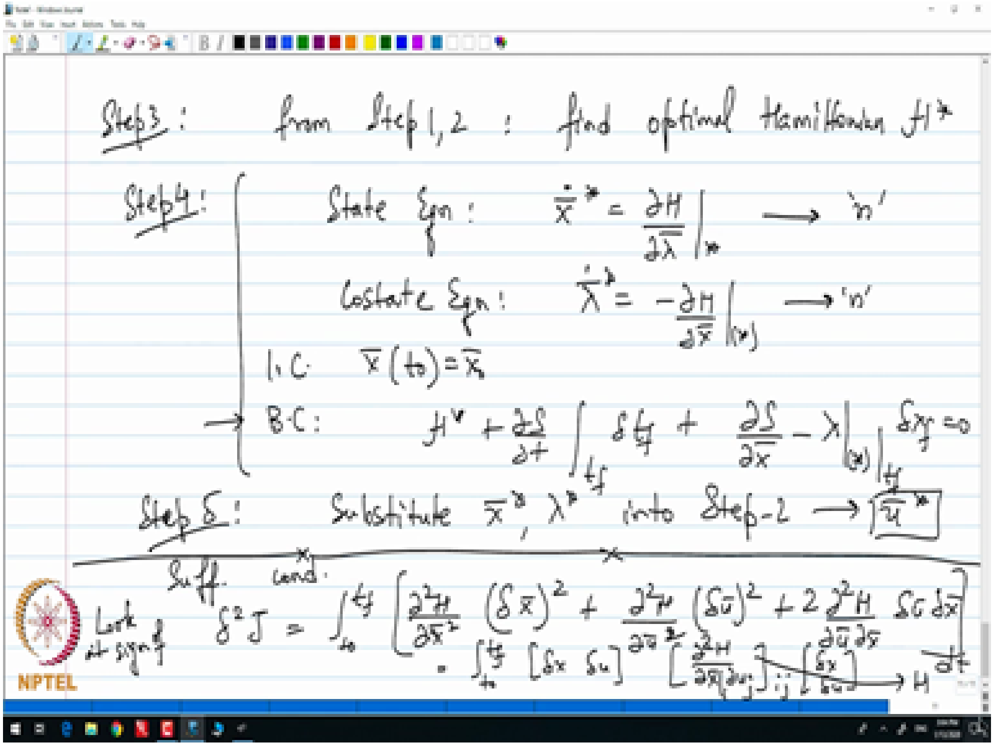

Step 3 : From step (1) and (2), find the optimal Hamiltonian $H^\star$ by substituting $u^\star$.

Step 4 : Once we have the optimal Hamiltonian we set up constrained equation, we have the state equation $\dot{\bar{x}}^\star = \frac{\partial H}{\partial \lambda}\mid_{\bar{x}^\star}$ and co-state equation $\dot{\lambda}^\star = -\frac{\partial H}{\partial \bar{x}}\mid_{\bar{x}^\star}$ and initial condition is $\bar{x}\left(t_0\right) = \bar{x_0}$ and boundary condition is given by

$$H^\star + \frac{\partial S}{\partial t}\mid_{t_f} \delta t_f + \frac{\partial S}{\partial \bar{x}} - \lambda \mid_{\bar{x}^\star}\mid_{t_f} \delta x_f = 0$$

And finally, after solving all $2n+1$ equations, notice that these are n vector equations, but first order and these are another n equations. So, we are solving $2n+1$ equations, the last one is this boundary condition.we have found all the variables the state and the co-state and then we plug it back into step (2).

Step 5 : Substitute $\bar{x}^\star, \lambda^\star$ into step (2) to get $\bar{u}^\star$, which is the optimal control.

One final point of discussion in the solution methodology is how about the sufficient condition? We have described the necessary condition for finding the optimal value. To find the sufficient condition we have to look at the sign of the second variation, but we are dealing with convex functions so the Hessian of the second variation will always be positive definite. The only thing that we have to check is a relation very similar to the strengthened Legendre condition. So, the sufficient condition is we have to look at the sign of second variation of

$$\delta^2 J = \int_{t_0}^{t_f} \left[\frac{\partial^2 H}{\partial \bar{x}^2}\left(\delta \bar{x}\right)^2 + \frac{\partial^2 H}{\partial \bar{u}^2}\left(\delta \bar{u}\right)^2 + 2\frac{\partial^2 H}{\partial \bar{u}\partial \bar{x}}\delta \bar{u}\delta \bar{x}\right] dt$$

we can write it in the form of the Hessian matrix.

$$\delta^2 J = \int_{t_0}^{t_f} \begin{bmatrix}\delta x & \delta u\end{bmatrix} \left[\frac{\partial^2 H}{\partial \bar{x}_i \partial \bar{u}_j}\right]_{ij} \begin{bmatrix}\delta x \\ \delta u\end{bmatrix}$$

So, I have written the above expression in a more compact notation, and let me call $\left[\frac{\partial^2 H}{\partial \bar{x}_i \partial \bar{u}_j}\right]$ as a matrix A.This is going to be positive definite, which is certainly the case.

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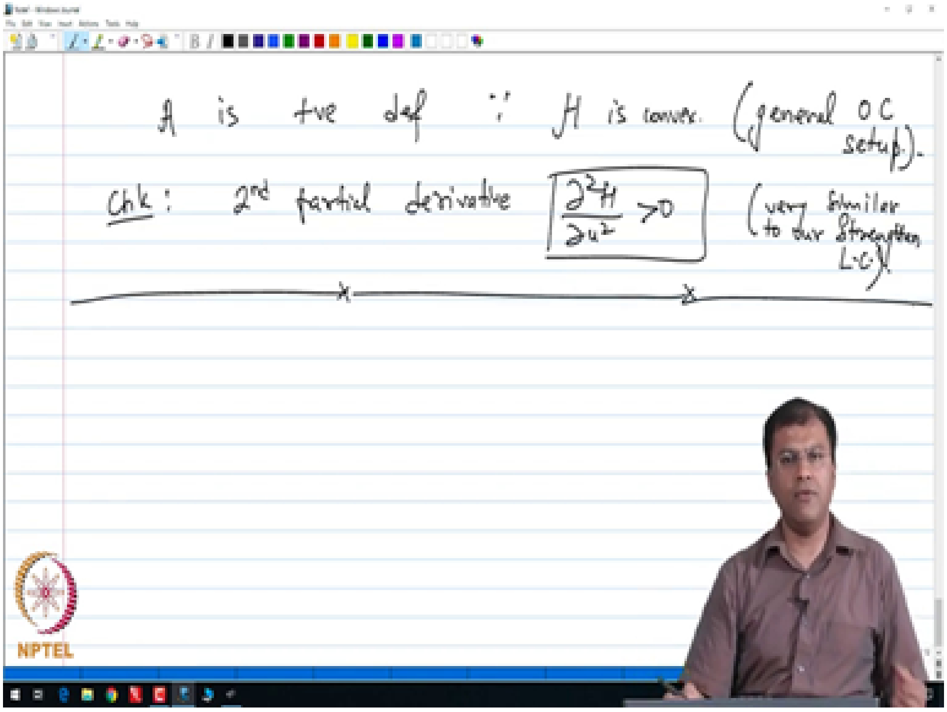

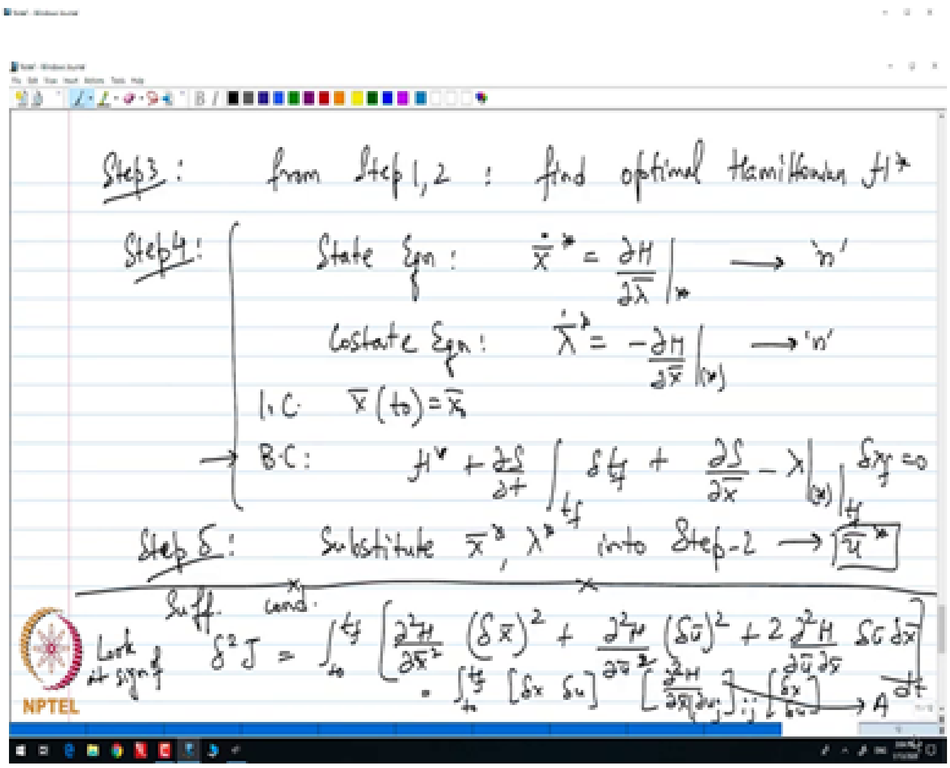

The matrix A is positive definite because our function H is convex. This is the general optimal control setup we take convex functions. And, the only thing that we have to check is the second partial derivative, given by $\frac{\partial^2 H}{\partial u^2} > 0$ .So, all we need to do is check whether the sign of this partial derivative is positive or not and this is very similar to strengthened Legendre condition.