Variational Calculus and its Applications in Control Theory and Nano mechanics
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Special cases / Invariance
Part-3

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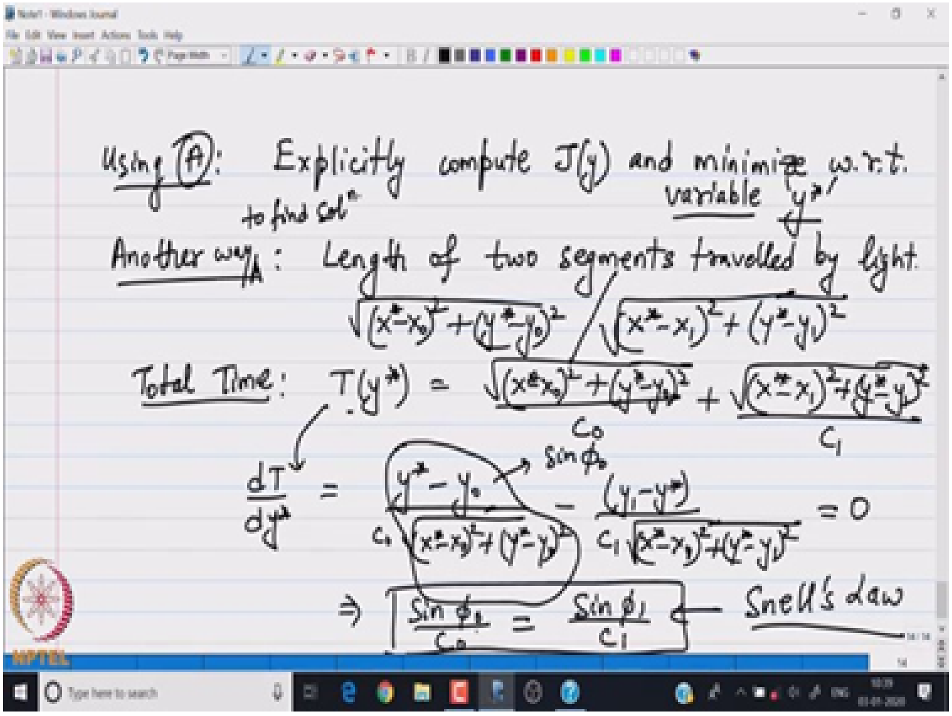

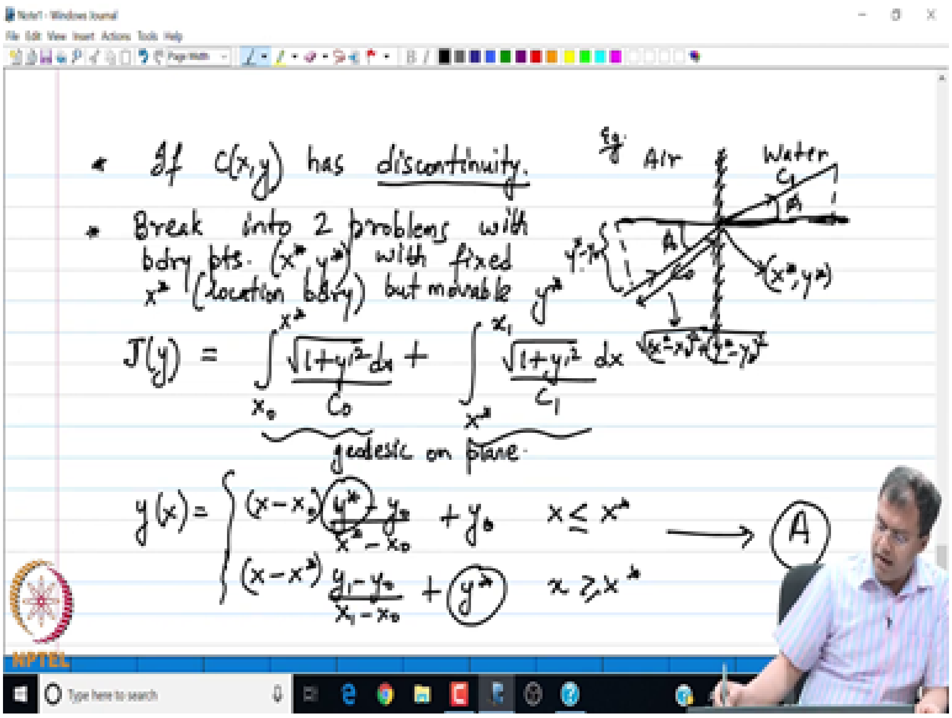

There is an even simpler method to do that rather than doing all this laborious work, we could look at this problem in another way and it is much more intuitive to look at this second way. So, let us see what is the length of the path taken by the light in each of the media.

Another way: Length of two segments along which light travels
$\sqrt{(x^* - x_o)^2 + (y^* - y_o)^2)}/\sqrt{(x^* - x_1)^2 + (y^* - y_1)^2)}$

$$TotalTime: \qquad T(y^*) = \frac{\sqrt{(x^* - x_o)^2 + (y^* - y_o)^2)}}{C_o} + \frac{\sqrt{(x^* - x_1)^2 + (y^* - y_1)^2)}}{C_1}$$

So, total length divided by the velocity in each media is going to give me the total time that the light particle takes from going from point A to point B.

So, now we have a function which is purely a one variable function and the one variable is y^*. So, now to find the minimum time we have to differentiate the T with respect to y^*

$$\frac{dT}{dy^*} = \frac{y^* - y_o}{C_o\sqrt{(x^* - x_o)^2 + (y^* - y_o)^2)}} - \frac{y_1 - y^*}{C_1\sqrt{(x^* - x_1)^2 + (y^* - y_1)^2)}} = 0$$

So, going back to our slide, next slide again, we see that the particular quantity here which I have circled is nothing but the sin of the angle of incidence, $\sin\phi_o$. So, what we get? Well, of course, to find the critical point we set this expression equal to 0 but this expression is nothing but the following expression

$$\Rightarrow \frac{\sin\phi_o}{C_o} = \frac{\sin\phi_1}{C_1} \qquad Snell's \;\; Law$$

Well, people who have done science in class 12th, they will immediately recognize that this is nothing but the famous Snell's law. So, in class 12th we are taught Snell's law we are just given the expression but we are not shown how we are, we get this expression. So, what this example shows that the time taken by the light is such that it always follows the Snell's law.

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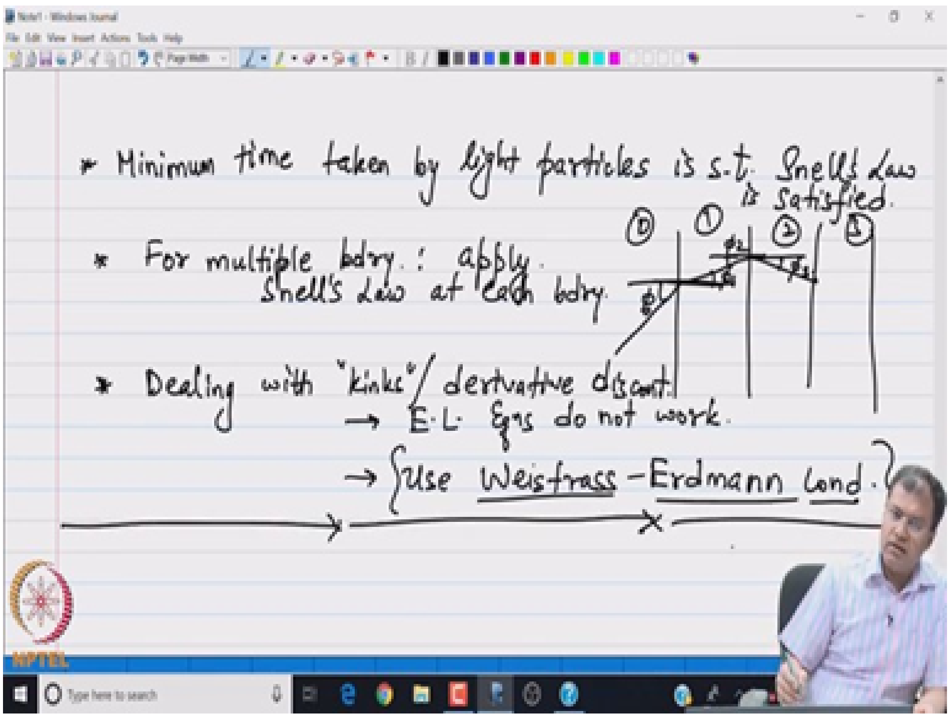

So, the moral of the story is the following, what we have is that the minimum time taken by light particle, the minimum time taken by the light particles is such that Snell's law is satisfied. So, people with a basic science background in high school, they all are familiar with Snell's law but they do not know how does it satisfy the time taken by the light particle. So, today in this lecture we have shown exactly that.

So, we can continue this discussion by looking at the various, we can extend this problem as to the path followed by the light in different, different medias, not only just one media but maybe several medias. So, we have media 1, let me call this as media 0, media 1, media 2, media 3 and what happens that in each of this medias, let me call this angle of incidence as ϕ_o, ϕ_1, ϕ_2 and ϕ_3.

So, in each of these mediums, so for multiple boundaries, the minimum will always be such that Snell's law is satisfied at each boundary. However, what we have seen is we have not exactly used the Euler-Lagrange equations in its purest form. We have broken down the problems into simpler bits and applied Euler-Lagrange equations in each of these bits wherever the solution is continuous, and continuously differentiable up to second order.

So, which means, well although this problem was fairly simple, in general dealing with kinks or derivative discontinuities is not very easy and in that case the Euler-Lagrange equations do not work, they do not work because of the underlying assumption of having second order continuous partial derivatives.

So, in that case we have to use a special result known as the Weierstrass-Erdmann condition of broken extremal which we will introduce in around the 10th lecture of this lecture discourse. So, we are going to deal with broken extremals in detail but right now I have just introduced with an example. So finally, we, let us now look at the fourth case of our special case of Euler-Lagrange.

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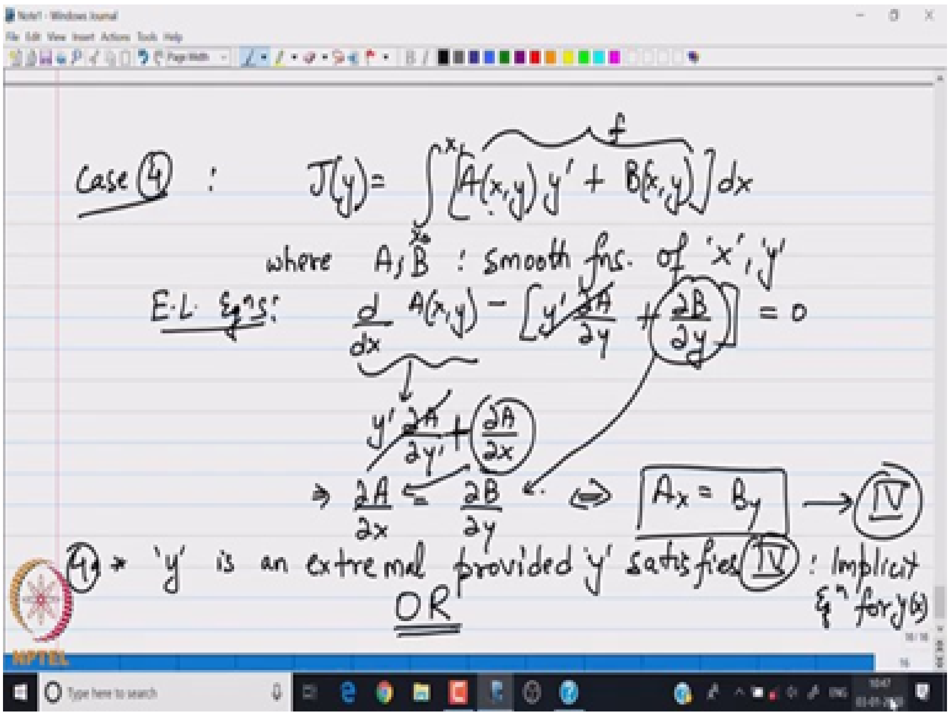

The fourth case is the case of degenerate solution. So, here I have a functional

$$J(y) = \int_{x_o}^{x_1} A(x,y)y' + B(x,y)dx$$

Where A, B are smooth function of x and y, otherwise we would not be able to apply Euler-Lagrange again.

If we apply Euler-Lagrange equations, we see that the equations reduces to the following

$$\frac{d}{dx}A(x,y) - \left[y^{'}\frac{\partial A}{\partial y} + \frac{\partial B}{\partial y}\right] = 0$$

$$\Rightarrow \qquad y^{'}\frac{\partial A}{\partial y} + \frac{\partial A}{\partial y} - \left[y^{'}\frac{\partial A}{\partial y} + \frac{\partial B}{\partial y}\right] = 0$$

$$\Rightarrow \qquad \frac{\partial A}{\partial y} = \frac{\partial B}{\partial y} \qquad \Leftrightarrow \qquad A_x = B_y \qquad \textbf{IV}$$

So, in this case, my extremal will be such that **IV** satisfied. So, I will not get y explicitly as a function of x, but I am going to get a relation which is satisfied by the extremal y. So there are two observations, so we say that y is an extremal, in this case y is an extremal provided y satisfies **IV**

So, this is an implicit equation for y(x), well this is the case when y needs to satisfy the relation **IV**. How about a case where this is trivially satisfied for all x and y? let me call this as case **4a** where if this holds then **IV** needs to be satisfied. We could also have another case.

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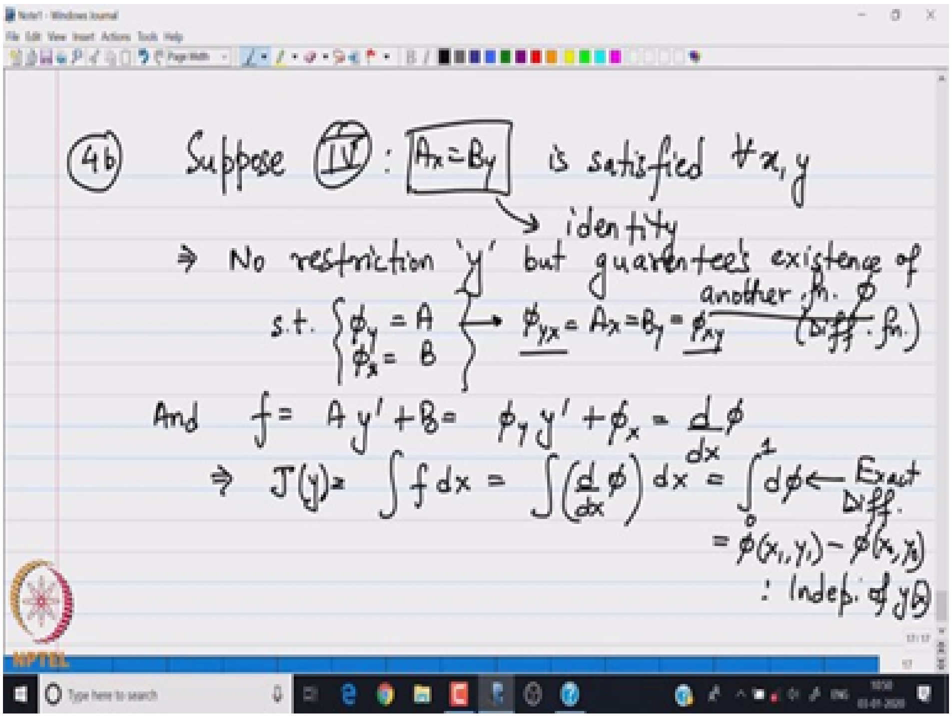

We could have another case, let me call this as **4b** where the boxed expression **IV** which is $A_x = B_y$ is satisfied for all x and y. So, which means that this particular expression is an identity as it is satisfied for all x and y, then this is an identity which means that there is no restriction on y. We just cannot find the extremal, however this particular set up guarantees the existence of another function ϕ, which is a differentiable function. It guarantees the existence of another function which is a differentiable function.

Such that $\phi_y = A$ and $\phi_x = B$. Why because, notice that if we were to take, assume this expression we see that $\phi_{yx} = A_x = B_y = \phi_{xy}$ So, if ϕ is differentiable then immediately this result is trivial, $\phi_{xy} = \phi_{yx}$. The mix derivatives are equal.

So, which means that if the boxed expression is an identity then there is another ϕ such that ϕ satisfies this particular set of two relations and
$f = Ay^{'} + B = \phi_y y^{'} + \phi_x = \frac{d}{dx}\phi$

$$\Rightarrow J(y) = \int f dx = \int \left(\frac{d}{dx}\phi\right) dx = \int_{\phi_o}^{\phi_1} d\phi = \phi(x_1, y_1) - \phi(x_o, y_o)$$

So, notice that the particular integral is an exact differential and notice that this particular quantity that we have found is independent of y(x). So, you just cannot find the extremal in this case because regardless of y, the functional is always going to reduce to this constant which only depends on the endpoints. So, now let us look at an example in this case.

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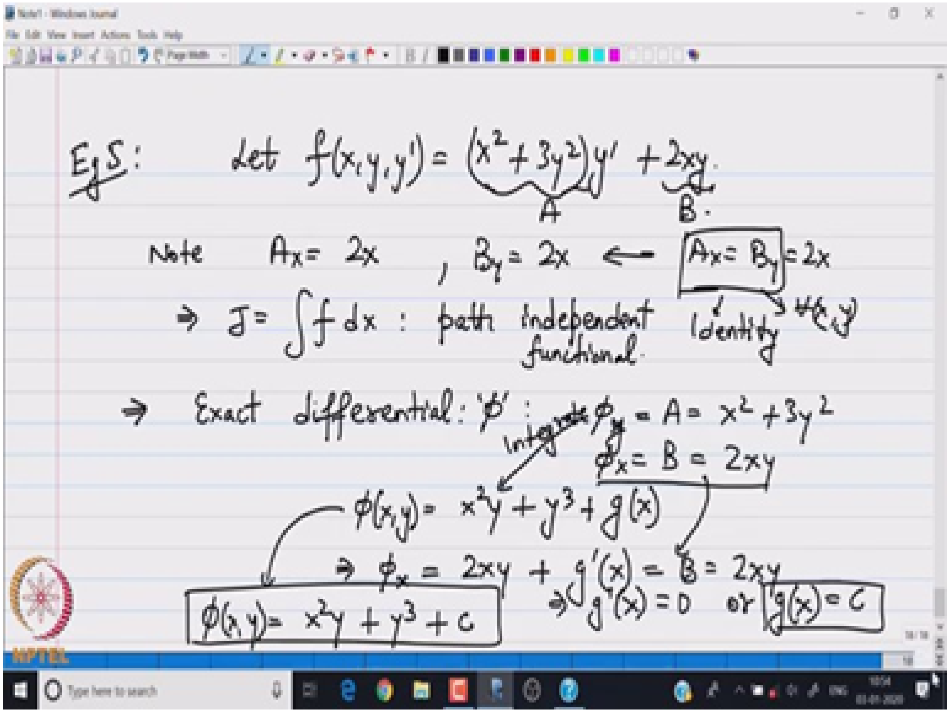

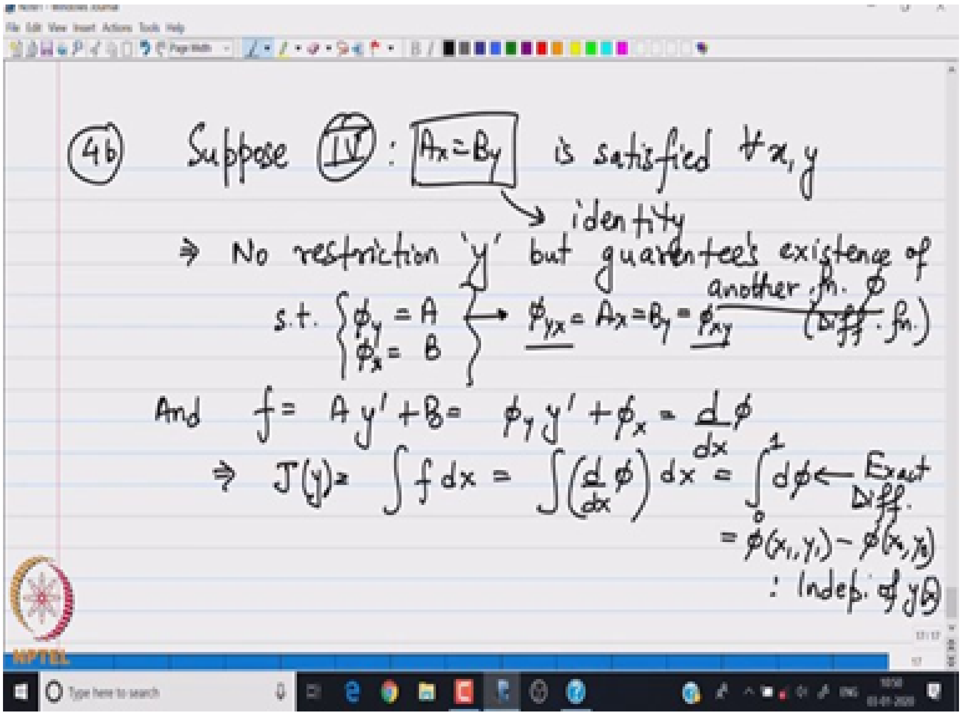

Example 5 Let $f(x, y, y^{'}) = (x^2 + 3y^2)y^{'} + 2xy$

Solution: $A_x = 2x, B_y = 2x \quad \Rightarrow \quad A_x = B_y = 2x$

So, we see that this is an identity, So, which means that my functional described by this function f is path independent functional and which means that it is exact, so all I need to do is to find the exact differential ϕ, the exact differential ϕ is given by ϕ_y.

The exact differential is $\phi_y = A = x^2 + 3y^2$ and $\phi_x = B = 2xy$

$$\phi(x, y) = x^2y + y^3 + g(x) \Rightarrow \phi_x = 2xy + g^{'}(x) = B = 2xy \Rightarrow g(x) = C$$

$$\Rightarrow \qquad \phi(x, y) = x^2y + y^3 + C$$

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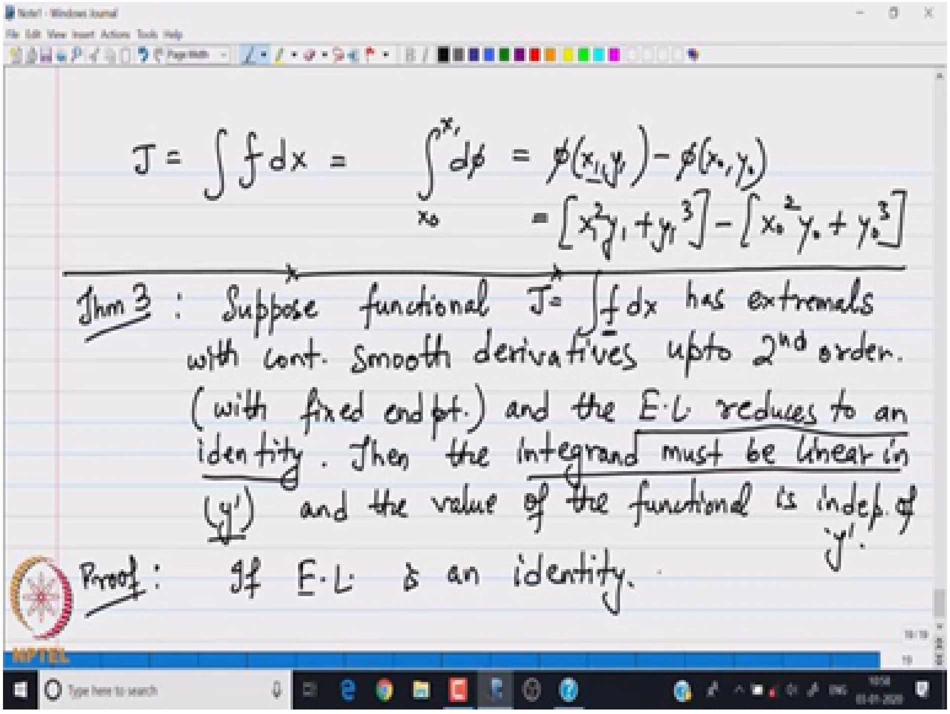

$J = \int_{x_o}^{x_1} d\phi = \phi(x_1, y_1) - \phi(x_o, y_o) = [x_1^2 y_1 + y_1^3] - [x_o^2 y_o + y_o^3]$

So, regardless of the function y, I am always going to get the value of the functional to be a constant which depends only on the final and the initial points. So, finding the extremal in this case is a useless exercise. So what I have done is the following, so let me rewrite this entire exercise in case 4 in the form of a theorem. Let me state the result in the form of a theorem.

Theorem 3 Suppose the functional J $J = \int f dx$ has extremals with continuous and smooth derivatives upto second order (with fixed endpoints) , this will be the final result in our lecture series.

So, we are still looking at fixed endpoint problems. And such that the Euler-Lagrange equation reduces to an identity, it reduces to an identity then what I have is that the integrand of the functional, the integrand f must be linear in $y^{'}$. And the value of the functional is independent of y.

So, what I have said in this result if the following, that whenever the Euler-Lagrange, this is the most important part of the statement, whenever Euler-Lagrange reduces to an identity it necessarily going to give us the case 4 that is the integrand will always be a linear function of $y^{'}$.

If Euler-Lagrange is an identity if and only if with the necessary and sufficient condition will lead to a case described by the case 4.

Proof : let us assume that the Euler-Lagrange equation is an identity.

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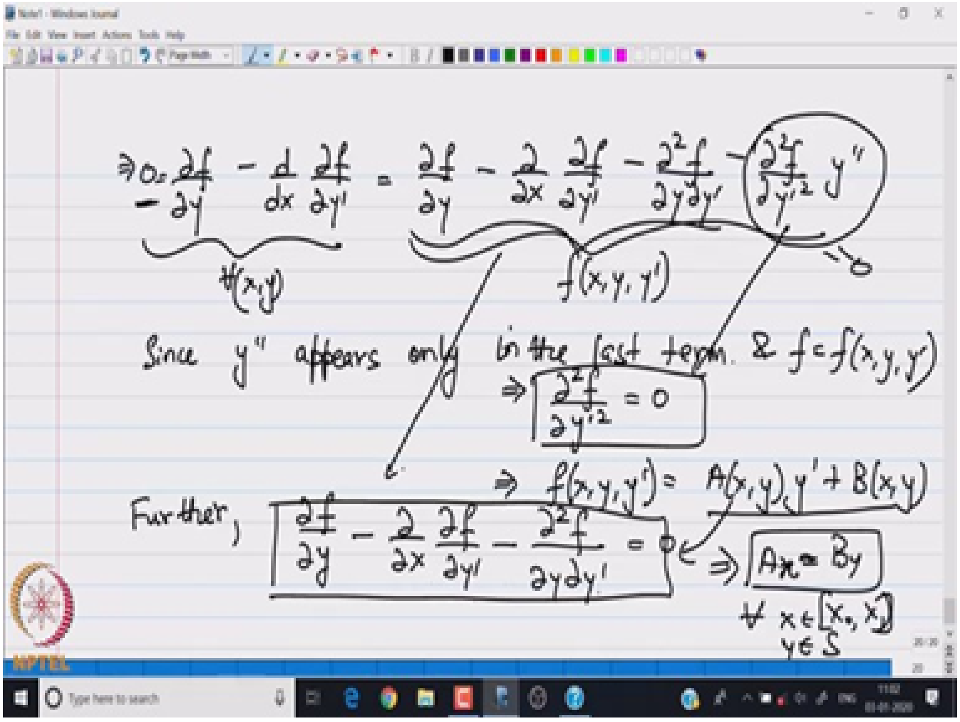

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y'} - \frac{\partial^2 f}{\partial y \partial y'} - \frac{\partial^2 f}{\partial {y'}^2} y^{''} = 0$$

Now, we are doing a problem in which f = f(x, y, $y^{'}$), so the maximum derivative argument that f has is only up to first order which means that the only term which contains the second derivative this falling quantity and since we do not have any terms involving second derivative of y.

So what I am saying is the following, since $y^{'}$ appears only in the last term and f = f(x, y, $y^{'}$) f which means that there would not be an explicit appearance of any terms involving $y^{''}$So, which means that $\frac{\partial^2 f}{\partial {y'}^2} y^{''} = 0$

So, from here I can directly deduce that f(x, y, $y^{'}$) is a linear function of $y^{'}$. and further $\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y'} - \frac{\partial^2 f}{\partial y \partial y'} - \frac{\partial^2 f}{\partial {y'}^2} y^{''} = 0 \Rightarrow A_x = B_y \quad \forall \quad x \in [x_0, x_1]$, for all y in the set of second order continuously differentiable functions satisfying the boundary condition.

So, the moral of the story is whenever we have the Euler-Lagrange equation being an identity, we always have that the integrand is a linear function of y and that the Euler-Lagrange equation finally reduces to this neat expression $A_x = B_y$.

So, thank you for listening. In the next lecture, I am going to talk about certain other topics related to Euler-Lagrange, namely the invariance, the existence uniqueness and further generalization of Euler-Lagrange equations. Thank you for listening. Thank you very much.