Basic Calculus - 1

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Lecture 7 - Part 2

One-sided Limits - Part 2

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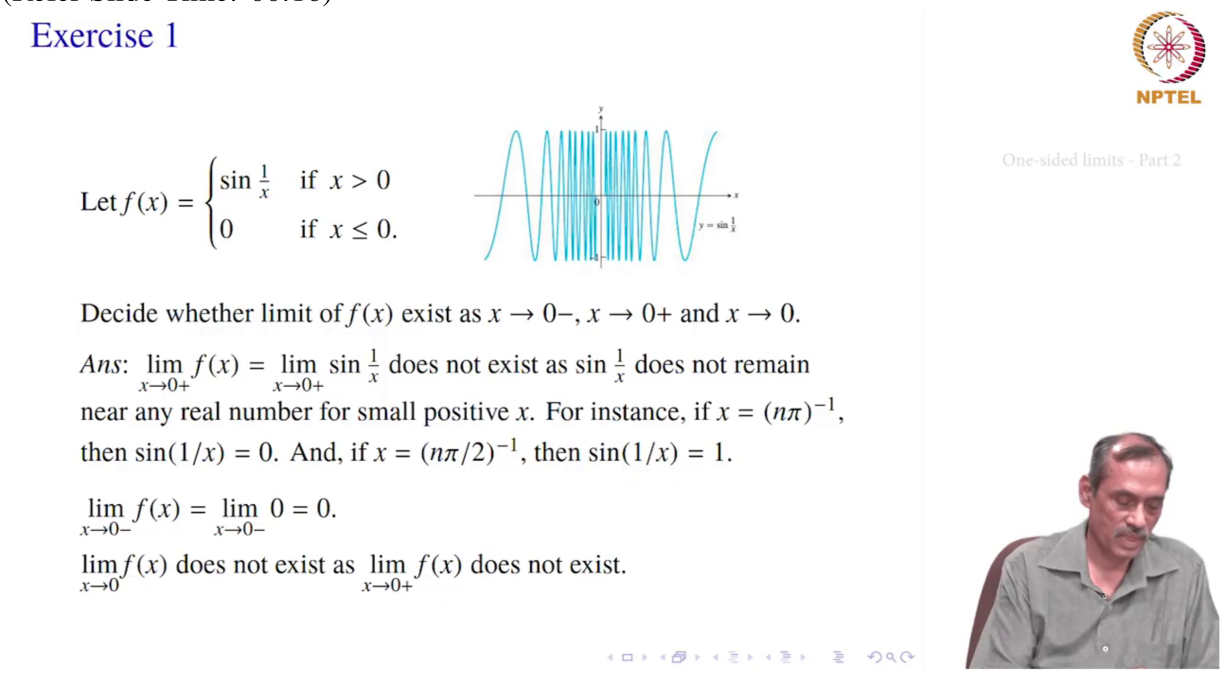

Let us take an exercise. Here, $f(x)$ is given as $\sin(1/x)$ if $x > 0$, and 0 if $x \le 0$. Recall that earlier we had considered $\sin(1/x)$. We know that the limit does not exist. It is $\sin(1/x)$; it looks like this. But now, what happens, this function is not exactly what is picturized. For $x > 0$, this side is all right. But on the left-side, for $x \le 0$, the function is only this. So, it is a different function. On the right-side, it is like this, and on left-side it is this way.

We want to find out whether the limit of $f(x)$ exists as x goes to $0-$, as x goes to $0+$, and as x goes to 0. That is, whether the left-side limit, the right-side limit, and the limit exist or not. What we are going to do is think of the first, say, the right-side limit. Now, limit of $f(x)$ as x goes to $0+$ is the limit of $\sin(1/x)$ as x goes to $0+$. And that we know, as earlier, it does not exist. Because, $\sin(1/x)$ does not remain near any real number. It varies between -1 and 1. You can give the earlier argument. Suppose x remains near 0. You can think of x as if it is $(n\pi)^{-1}$ for large n. If you find its limit, $\sin(1/x)$ for $x = (n\pi)^{-1}$ gives $\sin(n\pi)$, which gives you 0. But if you take x near 0 as the points like $(n\pi/2)^{-1}$, they are also near 0 for large n, then you would get $\sin(n\pi/2)$, which will be 1. That means, when x remains near 0, we do not get a limit. It neither stays near 0 always nor near 1 always. At these points, it does not remain near 0, at these points, it does not remain near 1. So, the limit does not exist. That is, the right-sided limit of this function at $x = 0$ does not exist.

What about the left-side limit? For the left-side limit for $x \le 0$, notice that the function is 0; it is a constant function. So, the limit exists because it is $f(x) = 0$. They are here. We are considering all points x which are near 0, but less than 0. At those points $f(x)$ is equal to 0. Therefore, the limit is equal to 0. So, the left-side limit exists and that is 0; the right-side limit does not exist. Therefore, limit of the function also does not exist because otherwise both the limits would have existed and being equal which is not the case.

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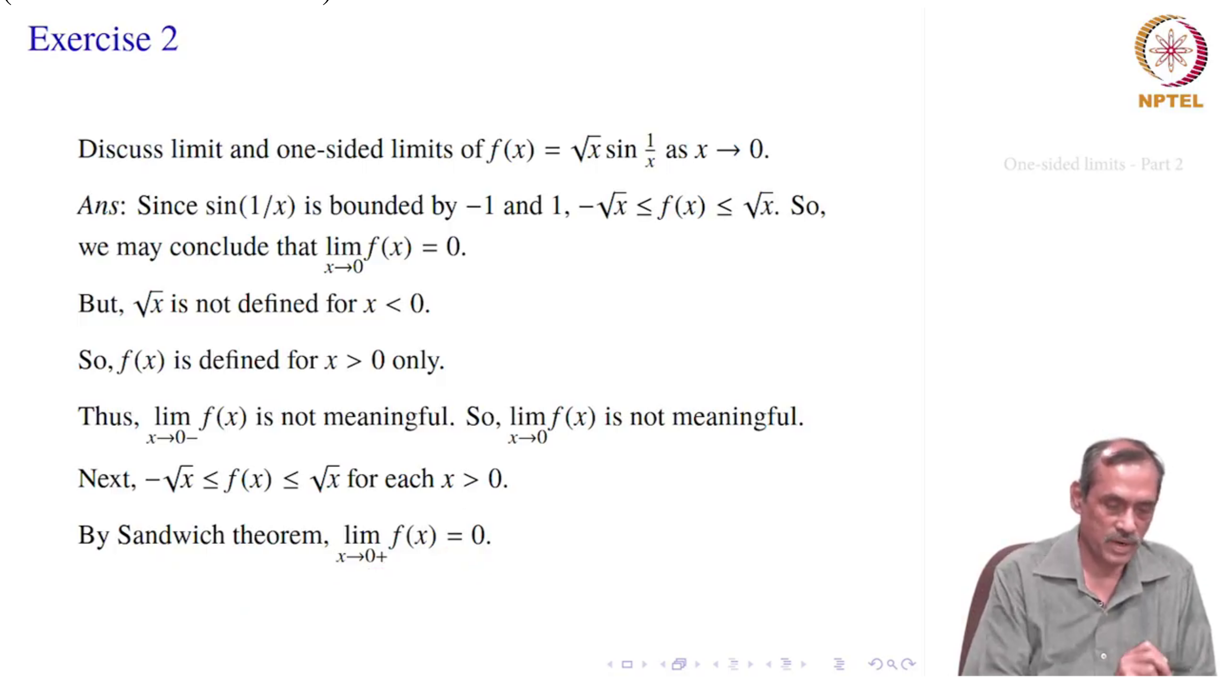

Let us take the next problem. Here we have this function. You had also seen this earlier. But we have to discuss something else; because we have more in our notions now. As earlier, $\sin(1/x)$ is bounded by -1 and 1. So, this function $f(x)$ is bounded by $-\sqrt{x}$ and $\sqrt{x}$. We may thus conclude that limit of $f(x)$ is 0. However, there is something else we have to consider. To see that the limit is meaningful at all, we should have a deleted neighborhood of $x = 0$ which should be contained in the domain of $f(x)$. And, here the domain of $f(x)$ does not include the negative numbers because $\sqrt{x}$ is not defined for that. So, the domain of $f(x)$ is only $x \ge 0$. At $x = 0$ again, $1/x$ is not defined. Therefore, the left-side limit is not meaningful because $\sqrt{x}$ is not defined for $x < 0$. It is defined only for $x > 0$. Therefore, the left-side limit is, you would rather say, not meaningful. Similarly, or because of that, the limit of $f(x)$ as x goes to 0 is not meaningful.

What about the earlier conclusion? We thought that it is bounded by this, and by Sandwich theorem it goes to 0. That gives a correct result but only whenever it is defined, whenever the concepts are meaningful. The same argument really gives that the right-side limit is 0. That is what you would get. But the limit does not exist.

So, you need also to consider whether these notions are at all applicable in a given case or not. In this case, the notion of left-side limit is not applicable at all; only the right-side limit is applicable, and that limit exists, and is equal to 0, for this function.

Let us take another exercise. We are required to evaluate as x goes to $0-$, that is, the left-side limit of the function $[\sqrt{6} - \sqrt{5x^2 + 11x - 6}]/x$. As earlier, the denominator now goes to 0. We have to see whether we can eliminate that possibility. So, what do we do?

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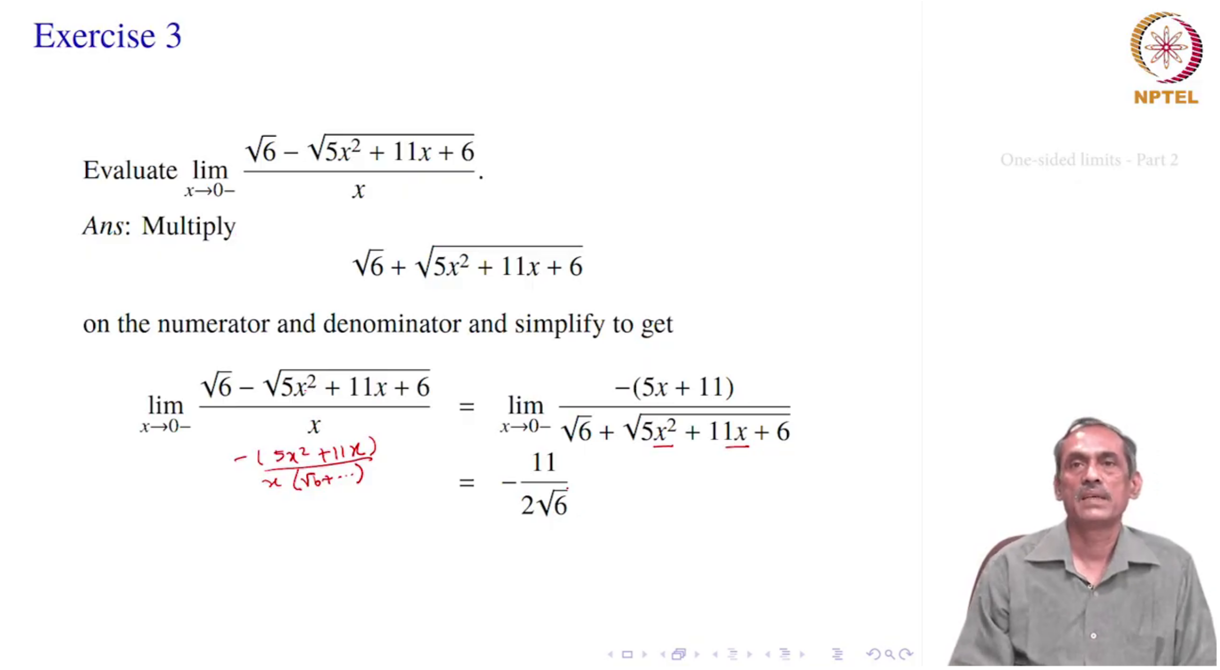

We rationalize that square; multiply it with $[\sqrt{6} + \sqrt{5x^2 + 11x - 6}]$ on the numerator and denominator. With that, the numerator becomes $6 - (5x^2 + 11x - 6)$. This 6 and the other 6 get canceled; you get $5x^2 + 11x$ on the top, and on the down side, you have $x[\sqrt{6} + \sqrt{5x^2 + 11x - 6}]$. Now, x gets canceled, and you get $5x + 11$ here. Of course, the minus sign remains. So, $5x + 11$, and it is minus sign here. It is $-(5x + 11)$ divided by the square root of 6 plus this expression.

Now the limit of the denominator does not go to 0. In fact that becomes $2\sqrt{6}$: it is $\sqrt{6}$, the other one becomes 0, so that it is $2\sqrt{6}$. And, the top becomes -11. So, the left-side limit of the function is $-11/(2\sqrt{6})$.

Let us consider the next problem. Here, you want to evaluate the limits and one-sided limits of this function. We have to consider three things: left-sided limit, right-sided limit and the limit if possible at $x = -1$. When x remains near -1, to what value this function remains near? That is the question.

Well, let us consider the left-side limit. The left-side limit means x will be negative. Let us take some neighborhood conveniently, say, from -1 to 0, where x varies. Then, this says $x + 1 > 0$; $|x + 1| = x + 1$. Because mod is there, we are doing this. As $|x + 1| = x + 1$, that cancels, and you get the function as $\sqrt{-2x}$. That is the function.

That means, when you consider the left-side limit of a $f(x)$, it is enough to consider $\sqrt{-2x}$. That is for the left-side limit. What will happen to the left-side limit then? It will be -2, x is remaining near -1, so that should be equal to $\sqrt{2}$. That is right as this is really minus sign and it is the left side limit.

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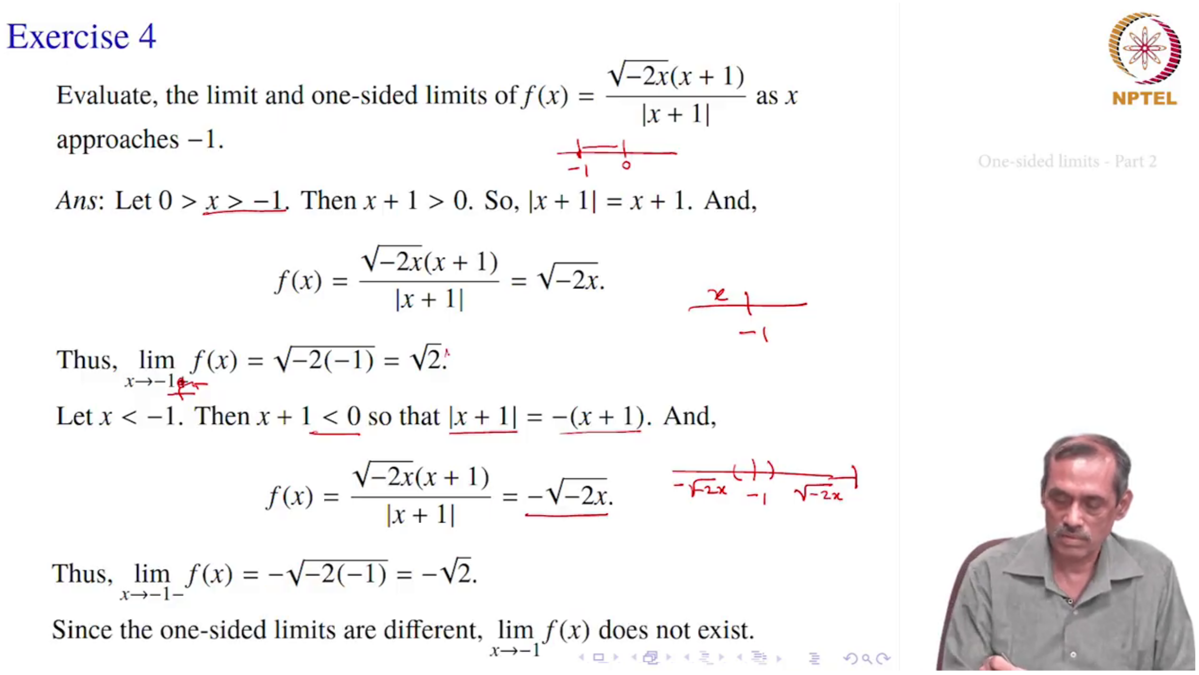

Now $x < -1$, or no, we have taken x to be bigger than -1; I see. It is not at the point 0, but at -1. You are concerned the limit at -1. So, you have considered a neighborhood of -1, this neighborhood. And, that is to the right of -1; so, this one must be correct; it extends to the left-side limit, x goes to -1 as $x > -1$. So, it was correct earlier that x goes to $-1+$. In that case, of course, the limit is $\sqrt{2}$.

Now, what about the left-side limit? Once we consider left-side limit at -1, not at 0, we are considering x to be here, $x < -1$. In that case, -1 goes, so $x + 1 < 0$. Since $x + 1 < 0$, $|x + 1| = -(x + 1)$. So, this is really $-(x + 1)$. Then, that factor $(x + 1)$ gets canceled, and we get $-\sqrt{-2x}$. So, what we are doing? If you take -1 here, on the left side, then $f(x)$ is $-\sqrt{-2x}$. On the right side, it is $\sqrt{2x}$. All the while x is remaining negative. So, we are not requiring, or we do not need even, to go beyond this 0. We need some neighborhood, even smaller than this is okay. But if you look at 0, it will be convenient to find the inequalities.

With this $f(x)$, we see that the right-side limit will be, as x goes to -1, that gives you -2 into -1, and its square root, so $-\sqrt{2}$. It will be again -1, and then it will be -2 times -1, but minus sign is outside; so that gives you $-\sqrt{2}$.

For this function $f(x)$ as x approaches -1, the left-side limit is $-\sqrt{2}$ and the right-side limit is $+\sqrt{2}$. Therefore, the limit does not exist when x goes to -1.

Let us see one more. We are required to find the limit of $\sin(3x)$ into $\cot(5x)$ divided by x times $\cot(4x)$ as x goes to 0. Again, the problem is the denominator becomes 0. But not so quickly; the function is not defined at $x = 0$. Let us see whether we can simplify it. In this expression, we can convert cot into sin and cos; that gives $\sin(3x)$ as it is, we write $\cot(5x)$ as $\cos(5x)$ divided by $\sin(5x)$. And on the denominator, we have $\cot(4x)$, which is $\cos(4x)$ on the denominator and to the numerator goes $\sin(4x)$. So, that is how it looks.

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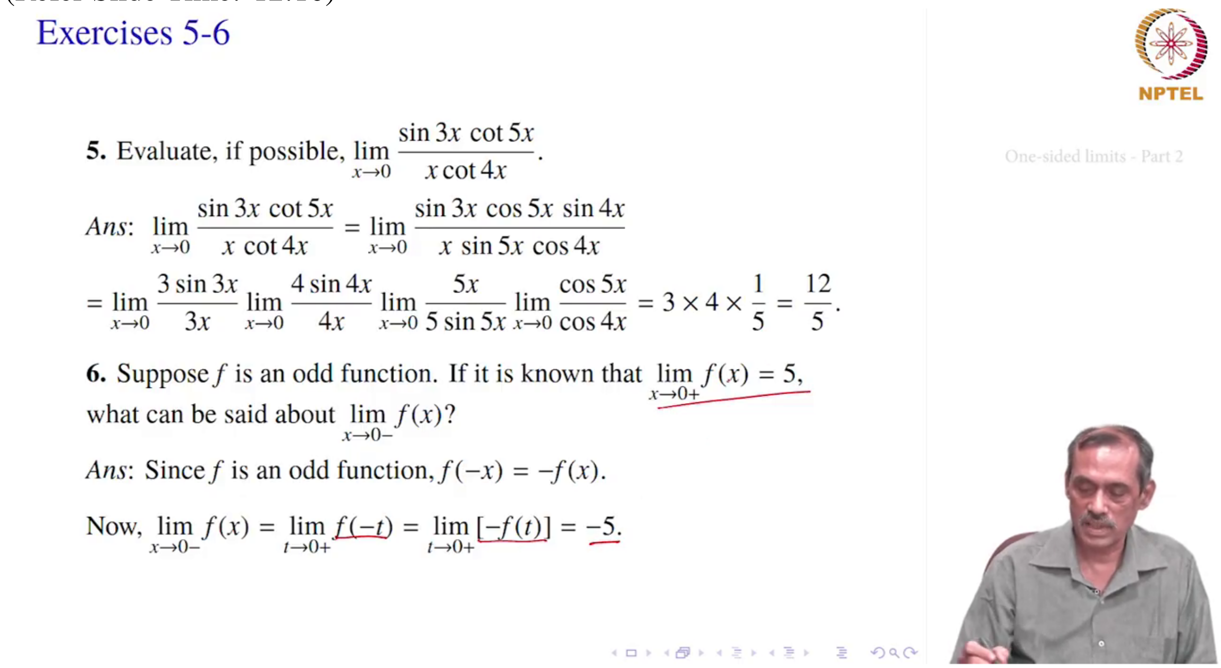

Our idea is to use our limit $(\sin x)/x$ when x goes to 0. To do that, we need $\sin(3x)$ by $3x$ because if that is the new variable t, then $t = 3x$. It will be in the form $\sin t$ by t. So, we write in that form, say, $\sin(3x)$ divided by x. We have $\sin(3x)$ by x. We are writing as $3\sin(3x)$ by $3x$, one 3 is multiplied on the numerator and denominator. Next, we look at $\sin(4x)$. For $\sin(4x)$, similarly, what do we do? We write $\sin(4x)$ by x first. Then, 4 gets canceled, $\sin(4x)$ by x, one x is divided, one x is multiplied. Here, $5x$, and it is $\sin(5x)$. So, this 5 is canceled, x divided by $\sin(5x)$, and the remaining is $\cos(5x)$; divided by $\cos(4x)$.

Now you can see whether it is same as the earlier: 3 gets canceled; 4 gets canceled, 5 gets canceled; next x gets canceled. So, you read on the top: $\sin(3x)$, $\sin(4x)$, $\sin(3x)$, $\sin(4x)$. On the bottom it is x times $\sin(5x)$, x times $\sin(5x)$, and the remaining is $\cos(5x)$ by $\cos(4x)$. Now, I think we can take the limit. This shows that $3sin(x)$ by $3x$, that is, 1; $\sin(3x)$ by $3x$, that is, 1; it is, if you take $3x = t$, then it gives limit as t goes to 0 of $\sin t$ by t, that is, 1. So the whole limit becomes 3. Similarly, your $\sin(4x)$ by $4x$, that goes to 1; so it is 4. Similarly, here it becomes $5\sin(5x)$ by $\sin(5x)$; that again goes 1; so, you get $1/5$. And, $\cos(5x)$ goes to 1; we know $\cos x$ as x goes to 0 is 1, so, $\cos(4x)$ also goes to 1. So, that is just 1. You get 3 into 4 into by 5, or $12/5$.

So, some rewriting of this sort may be required in some of the problems.

Let us go to next problem. Here, we are asked not to just to evaluate the limit but something else using the properties of functions. Suppose it is given that $f(x)$ is an odd function, that is, $f(-x) = -f(x)$. It is known that the right-side limit of $f(x)$ when x is near 0, is equal to 5. So, can we do something about the left-side limit of $f(x)$ as x goes to 0? That is the question.

Well, we have to obviously use the property of an odd function. Now, f is an odd function; so, $f(-x) = -f(x)$. When you take $f(x)$ as x goes to $0-$, then that is the same thing as limit of $f(-t)$ as t goes to $0+$. Then we can write $f(-t)$ as $-f(t)$ because it is odd. So, you are taking this negative sign out. You are finding the limit of $f(t)$ as t goes to $0+$, which is given as 5. Therefore, the answer is -5.

For an even function, something else might happen. This result may not be correct. Because it is even you get $f(-x) = f(x)$. Then, this one becomes $f(x)$. There is no minus sign. It will be the same as that. So, let us stop here.