Basic Calculus - 1
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Lecture 30 - Part 2
Area between Curves - Part 2

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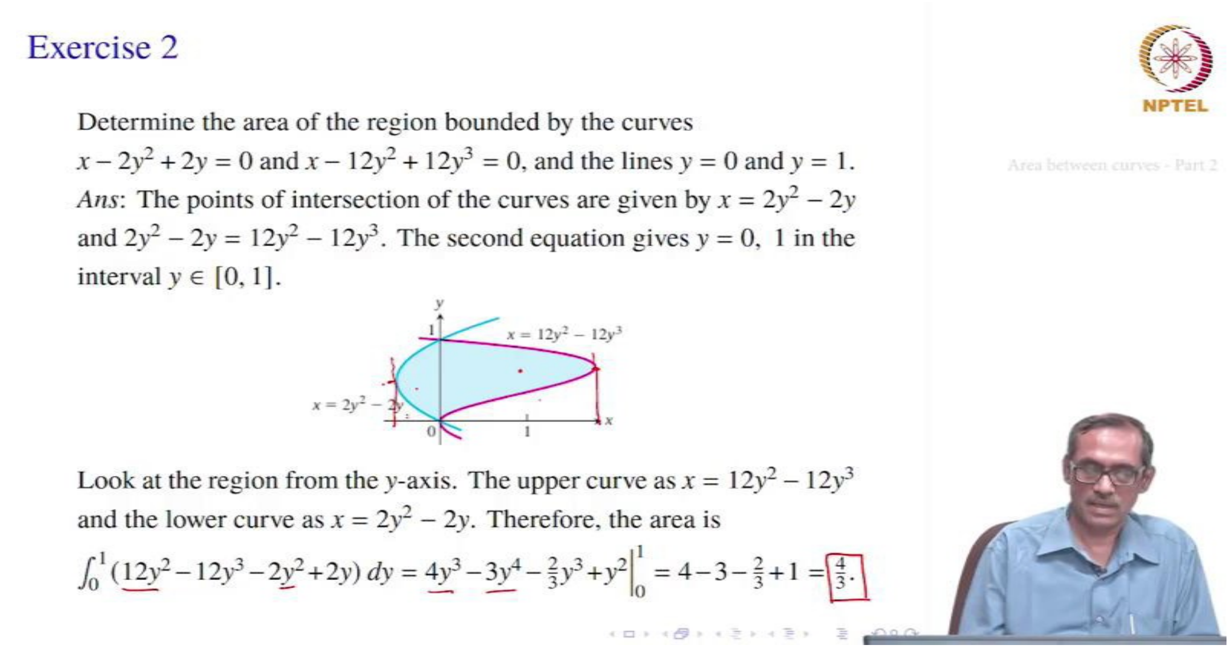

Let us take another problem. You want to determine the area of the region bounded by the curves $x - 2y^2 + 2y = 0$ and $x - 12y^2 + 12y^3 = 0$, and the lines $y = 0$ and $y = 1$. The two curves are given implicitly here. It is like whatever we have started with in the beginning; it looks something like that. First, we have to see how the curve look like; whether their points of intersection are falling exactly on those two lines or something else.

Let us find the points of intersection of the curves. We can write the first curve as $x = 2y^2 - 2y$ and the second as $x = 12y^2 - 12y^3$. Eliminating x we get $2y^2 - 2y = 12y^2 - 12y^3$. We now solve for y to get $y = 0$ or $y = 1$. The other value of y does not lie between 0 and 1, and all that we want is in the interval $[0, 1]$. Beyond that there can be roots, but we are not worried about that. So, in this interval $[0, 1]$, we have only two solutions of this, where $y = 0$ and $y = 1$. If you really plot these curves, it would look something like this: the curve on the left, the blue one is $x - 2y^2 + 2y = 0$; and the magenta one is $x - 12y^2 + 12y^3 = 0$.

You want to compute the area between them, where y varies from 0 to 1. These points are exactly the points of intersection. So, giving those lines is really superfluous here. One could have just asked “determine the area of the region bounded by these two curves that lie in the upper half plane”. Notice that we are getting the same region in the upper half plane only.

You can get this area in many ways. One is: you can divide that into sub-regions, if you want to integrate through x. But here x is given as a function of y. So, it is quite natural to go the other way around. If you look at it from the y-axis, the pink one is on the top and the blue curve is on the bottom. Then, you can write the area as the integral $\int_0^1 (12y^2 - 12y^3 - 2y^2 + 2y)\, dy = 0$. This is to be integrated. We know that integration of $3y^2$ gives us y^3, $4y^3$ gives y^4, and $2y$ gives y^2. So, we get $4y^3 - 3y^4 - (2/3)y^3 - y^2$. We evaluate this at 0 and 1, and subtract to we get finally the answer as $4/3$.

You see, looking at from y axis made it easier. If you do it from x-axis, you have to find this point and then subtract these two areas to get this area; and on the other side, find this point, and subtract this and this area, to get this area the whole and minus this one. That is how we can do it; but that will be a bit complicated.

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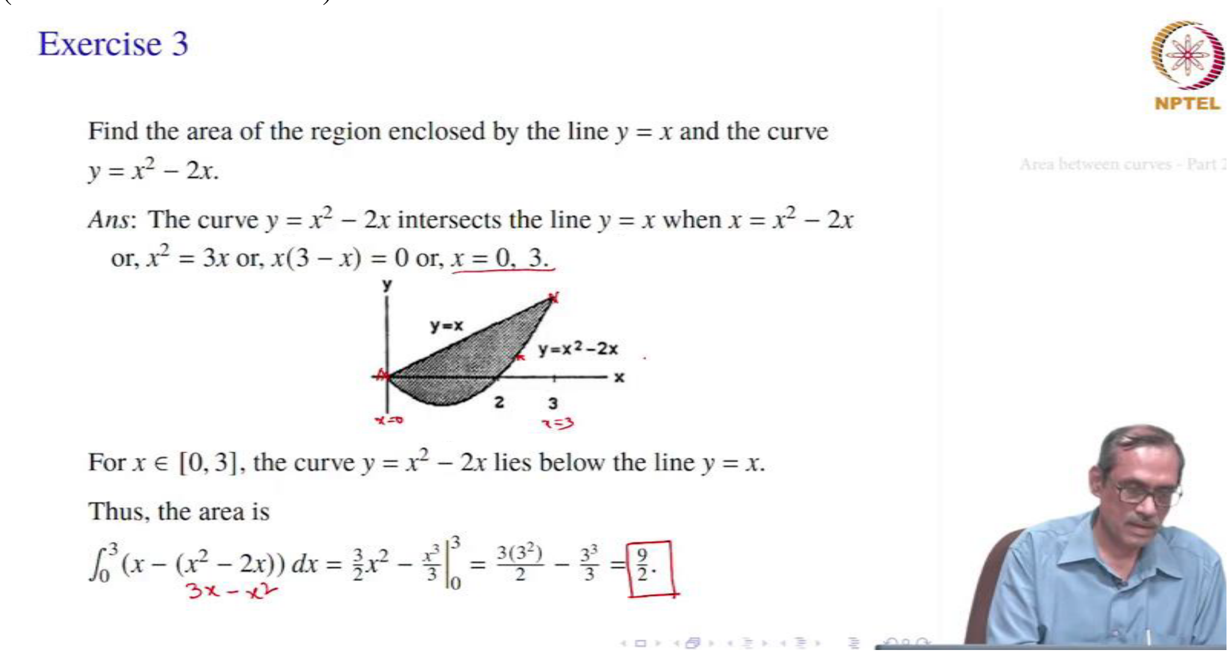

Let us take the next problem. Find the area of the region enclosed by the line $y = x$ and the curve $y = x^2 - 2x$. Since it is enclosed by these two, it looks that $y = x$ should intersect that at least at two points. And then some region should be formed. This is how it looks. You take $y = x$ and then $y = x^2 - 2x$, which is on the downside. We have this region then.

To get exactly what are those points where they intersect, we are going to solve the equations. We have $x^2 - 2x = y = x$. So $x^2 - 2x = x$. It has solutions as $x = 0$ or $x = 3$. You get the point $x = 0$ here, and $x = 3$ here. The line $y = x$ lies on the top of the curve $y = x^2 - 2x$. So, it will be fairly straightforward to use the integral. The area is the integral $\int_0^3 (x - x^2 + 2x)\, dx$. It gives $\int_0^3 (3x - x^2)\, dx$. The integral of $3x$ is $(3/2)x^2$ and the integral of $-x^2$ is $-x^3/3$. This gives the answer as $9/2$. This is fairly straightforward, but you have to find the points of intersection first.

Let us go to Exercise 4. We want to determine the total area of the regions bounded by the line $5y = x + 6$ and the curve $y = \sqrt{|x|}$. This is the square root of mod x. That means it is equal to $\sqrt{x}$ when $x \geq 0$ and it is $\sqrt{-x}$ when $x < 0$. This is the definition of $|x|$.

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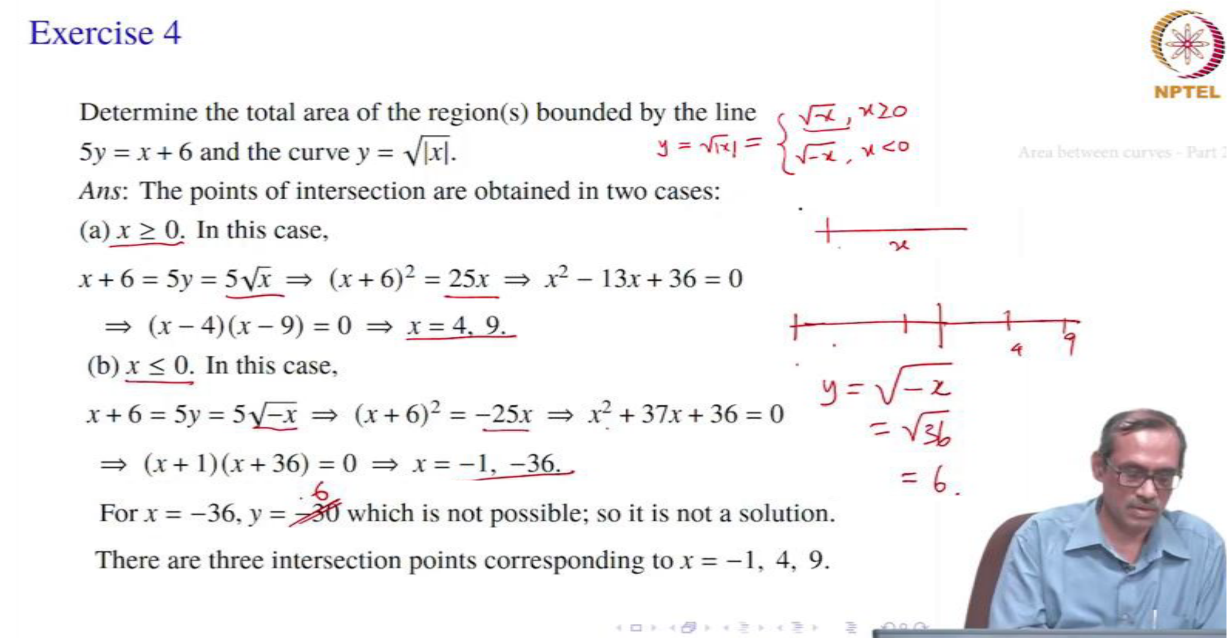

That is how the curve looks like. We have to consider at least two cases: when $x \geq 0$ and when $x < 0$, and then find the corresponding regions and add them up. Let us take the first case: $x \geq 0$. In this case, you want to find the points of intersection. The curves are $5y = x + 6$ and $y = \sqrt{x}$. So, $x + 6 = 5y = 5\sqrt{x}$. Eliminating y, we get $(x + 6)^2 = 25x$. That gives $x^2 - 13x + 36 = 0$. If you factorize, that would give $(x - 4)(x - 9) = 0$. So, in this case, that is, for $x \geq 0$, the points of intersection are $x = 4$ and $x = 9$.

Similar thing might happen again on the negative side, that is, when $x < 0$. Let us see that case. When $x < 0$ (or evne $x \leq 0$), we have $x + 6 = 5y = 5\sqrt{-x}$. It gives $(x + 6)^2 = 25(-x)$ or, $x^2 + 37x + 36 = 0$. Again, we factor it to get $(x + 1)(x + 36) = 0$. That gives two solutions $x = -1$ and $x = -36$. That means, when $x < 0$, the curves intersect at $x = -1$ and $x = -36$. Corresponding to these there will be points on the curve.

Look at the first case. The intersection points correspond to $x = 4$ and $x = 9$. There will be corresponding points on the curves. Now, you have to really find all the areas and then add them up.

Now, we just look at something. If $x = -36$, then $y = \sqrt{|x|} = \sqrt{36} = 6$. That is, $(-36, 6)$ is an intersection point. But this is not possible. Why is it not possible? What is the reason? We will see from the plot. What happens is, if x is negative, then $(-36, 6)$ does not satisfy $x + 6 = 5y$ as $(-36) + 6 \neq 5 \times 6$. That is, $(-36, 6)$ is really a false point, which is not falling on the straight line. And all the other points satisfy the equations. So, we get three intersection points corresponding to $x = -1$, 4 and 9. They are actually the intersection points. not the fourth one. $x = -36$ is a possible solution, but it is not, because for $x < 0$, $y = 6$ is not possible.

We want to determine the area bounded by these two lines. There is no region corresponding to this binding the area, that is why it is not possible. Only possible points are these, where the regions are formed. Now, these are the three intersection points, which correspond to $x = -1, 4, 9$.

These are shown in the picture. We compute the area which is painted black here; that is the area. But the other area is also included, which is looking very small in the picture. But there is some area there. It is not a straight line. Had it been a straight line, its area would have been 0. It is not, it is really intersecting at two points, and there is some region formed.

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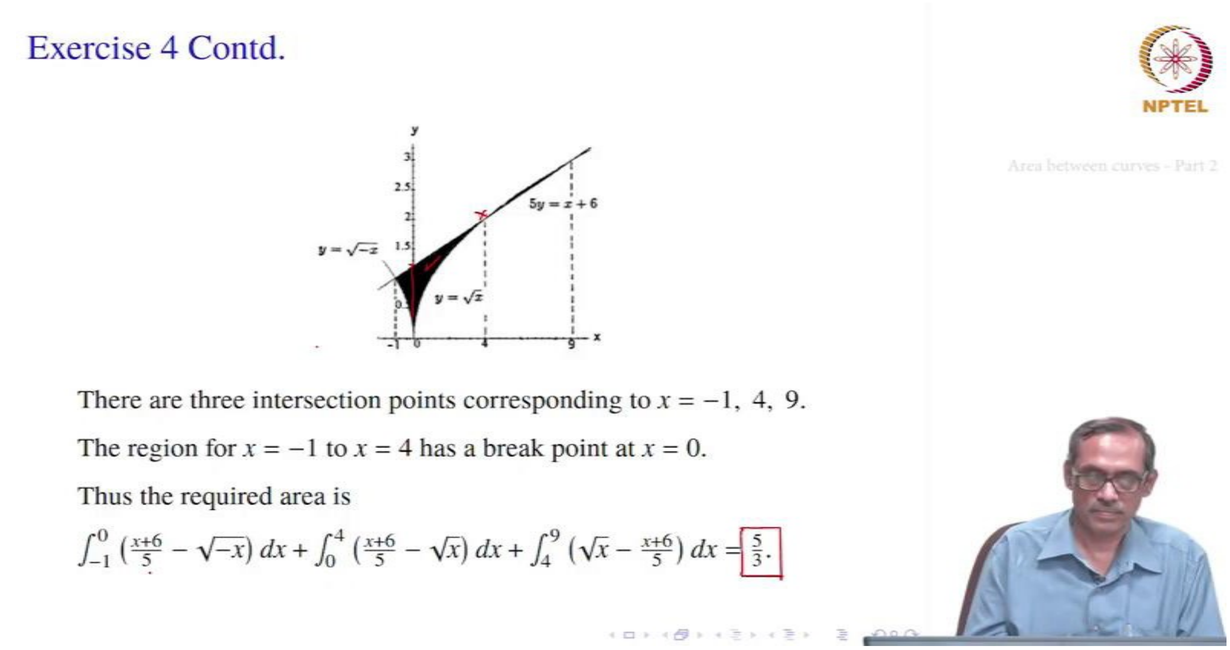

Now how do we compute the area of these regions? Of course, computing that from 4 to 9 will be straight forward. The top curve is the straight line and the bottom one is the curve. We can just integrate it. But this one, the bigger black region has to be broken out again. Let us say we break it at $x = 0$. Then we have an area which has on the top the straight line and on the bottom this curve. On the other side, again, you have one on the top and one on the bottom. You have really got three integrals that way. That is how we do it.

We break at $x = 0$ and write it as three integrals. One is from -1 to 0, where the line is $5y = x+6$ or $y = (x+6)/5$, which lies on the top side, and on the bottom we have the curve $y = \sqrt{-x}$ since here, $x \le 0$. Here x varies from -1 to 0.

And, for the other integral, we have this area, where the straight line is on the top and $y = \sqrt{x}$ on the bottom. Here, x varies from 0 to 4.

The third one has the straight line which lies below the curve. Since $x > 0$, here the curve is $y = \sqrt{x}$ and below it is the straight line $y = (x+6)/5$. Here, x varies from 4 to 9.

That is how we got the area written as sum of these three integrals. If you integrate them and simplify, you get the answer as $5/3$.

Sometimes plotting may become necessary to find out what are the possible solutions, and what are the imaginary ones which do not form regions.

Let us go to the next problem. Find the area of the region enclosed by $y = \sec^2(\pi x/3)$, $y = x^{1/3}$, $x = -1$ and $x = 1$. These lines $x = -1$ and $x = 1$ are two vertical lines. These two are curves and these two are lines.

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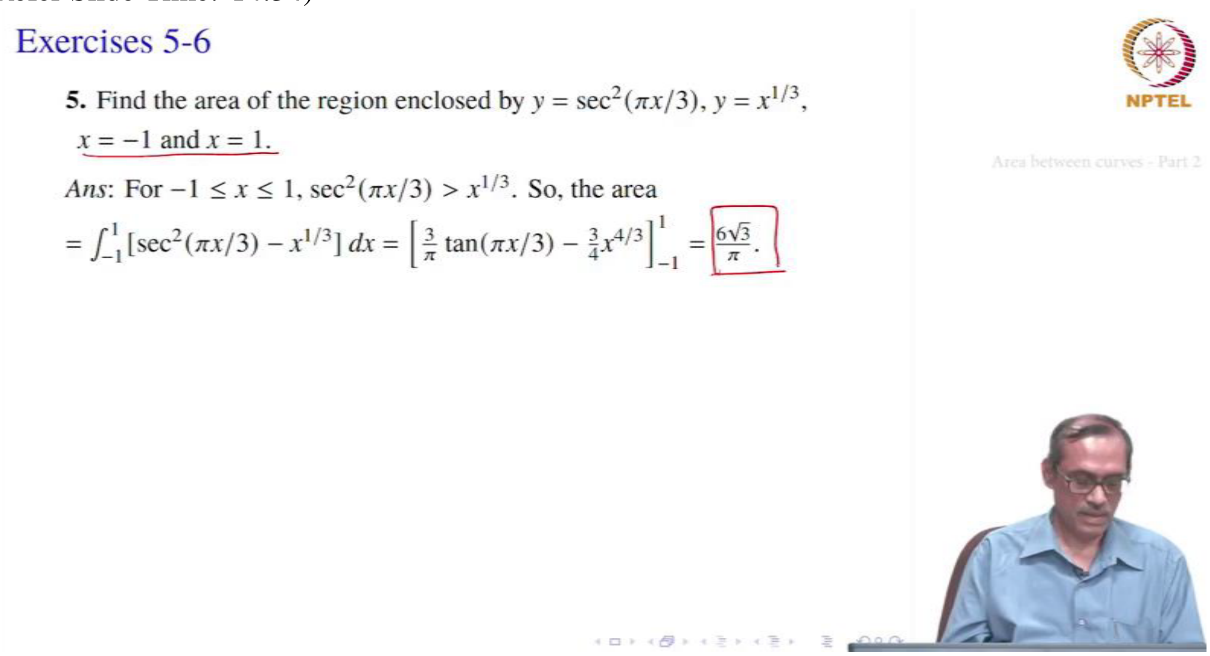

All that we have to find out is, which one is lying above what? And that would give us a straightforward answer. Here, x varies from -1 to 1. For these values of x, we see that $\sec^2(\pi x/3) \geq x^{1/3}$. Since it is bigger, we have the area as the integral $\int_{-1}^{1} \left[\sec^2(\pi x/3) - x^{1/3}\right] dx$. You integrate $\sec^2$ directly. For $\pi x/3$ you multiply $3/\pi$ etc. The integral of $x^{1/3}$ is $x^{4/3}/(4/3)$. This expression is to be evaluated at 1 and -1, then subtracted. That simplifies to $6\sqrt{3}/\pi$.

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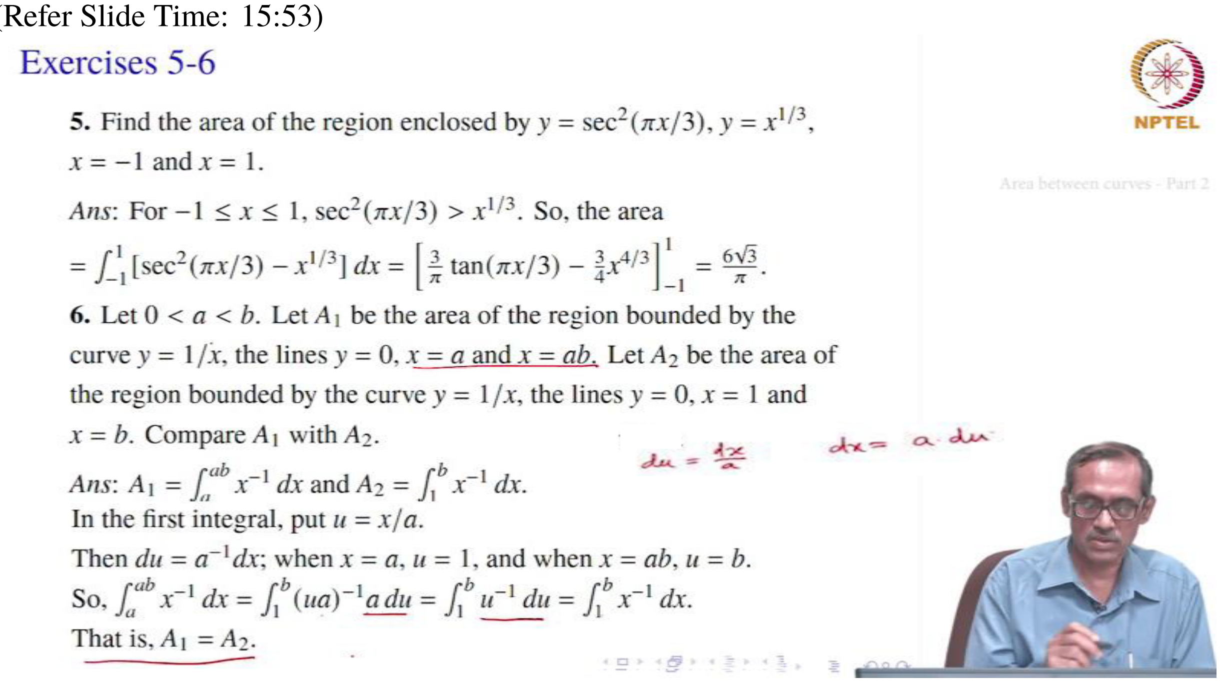

Let us solve another. Let $0 < a < b$. There are two positive real numbers a and b, where $a < b$. A_1 is the area of the region bounded by the curve $y = 1/x$ and the lines $y = 0$, $x = a$ and $x = ab$. There is one curve and three lines, two of which are vertical lines. Similarly, A_2 is the area of region bounded by the same curve, the x-axis, but now, the two vertical lines are different. They are $x = 1$ ad $x = b$. We are required to find out which area is bigger than what?

Let us write them as integrals. First, $A_1 = \int_a^{ab} [x^{-1} - 0]\, dx$ since x varies between a and ab, the curve $y = x^{-1}$ lies above the x-axis, that is $y = 0$. Similarly, $A_2 = \int_1^b [x^{-1} - 0]\, dx$. We have to compare these two integrals and say which one is bigger or what happens.

Since the limits are a and ab, this a is multiplied here. We have to put a substitution. Let us put the substitution $u = x/a$. We may also write it as $x = au$. Here, x is varying between a and ab. Since $u = x/a$, when $x = a$, $u = 1$; and when $x = ab$, $u = b$; so, u varies between 1 and b. In that case, what happens to the differential? You find that $du = d(x/a) = dx/a$. That is what we wrote here; or, you can write $dx = a \times du$. Using that we get

$$\int_a^{ab} x^{-1}\, dx = \int_1^b (au)^{-1} a\, du = \int_1^b u^{-1}\, du.$$

The last integral can also be written as $\int_1^b x^{-1}\, dx$ since it does not matter which variable it is in the integral. So, these two areas are same; that is what it says. Let us stop here.