Mathematical Methods 2
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Complex Numbers
Lecture - 01
Introduction to complex numbers

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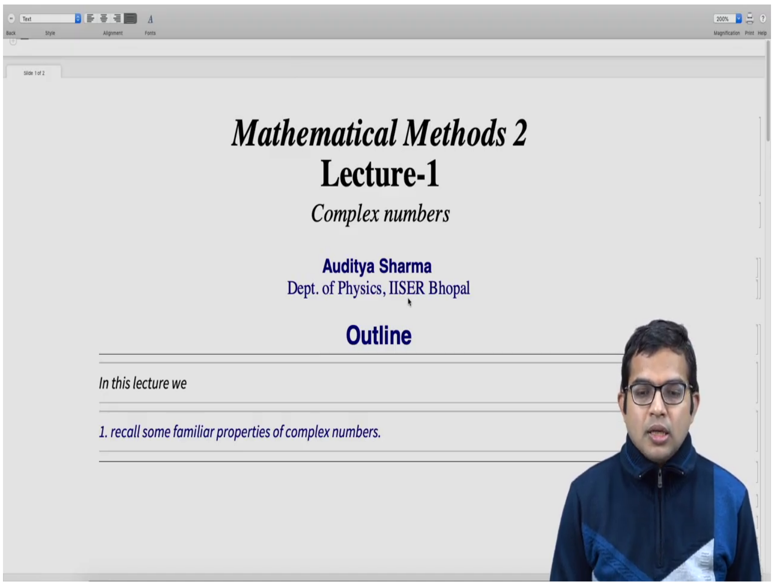

So welcome to Math Methods 2. So, we are going to use a strategy similar to Math Methods 1 where we broke down the overall material in lots of small units. So, for ease of organization both from our point of view and also for ease of you know study from the students point of view.

So, the first topic that we are going to cover is going to be about complex analysis in which we will begin with just Complex Numbers in this lecture and maybe a few more lectures we will just recall properties of complex numbers that we are all you know mostly expected to be familiar with. So, in this lecture we will just start with some very basic definitions. That is what this lecture is about.

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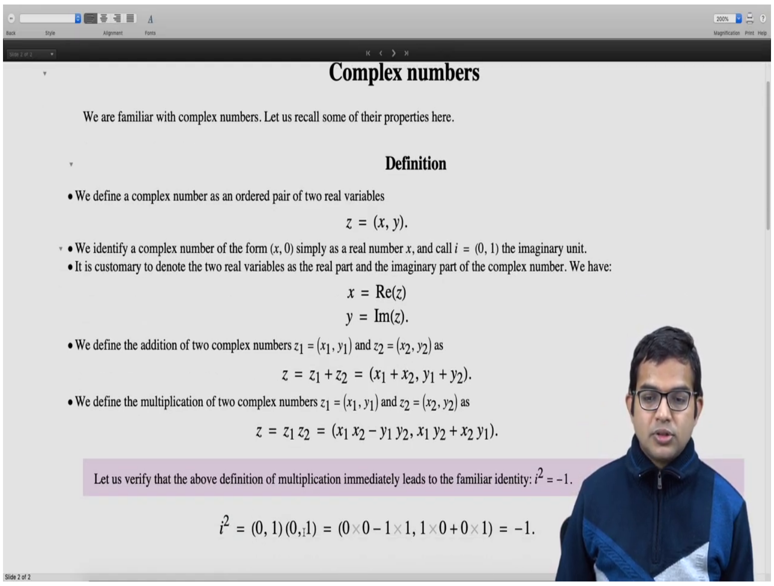

So, we can think of a complex number as essentially an ordered pair of 2 real variables right. So, there is what we call x and what we call y, also known as the real part and the imaginary parts. So, although this is a somewhat more abstract way of introducing the notion of a complex number compared to how we might have first encountered it.

So, usually we come across a complex number when we are trying to solve for certain algebraic equations and we see that within the real you know set it is not possible to find solutions for certain kinds of algebraic equations. And so the theory you know tells us that algebraic equations are guaranteed to have solutions if you go to complex numbers.

We first start with any natural numbers then we extend to include negative numbers, because certain equations do not have solutions with positive numbers then we extend it to real numbers and finally to complex numbers and that is it you do not need to go beyond complex numbers that is what the theory is.

But it is also possible to think of a complex number as you know is made up of two real numbers and it is an ordered set. So, it is important to identify one of them as a real part and the other one is the imaginary part. And so yeah so a number of the form (x, 0) if it has no imaginary part then that is just a real number that we are more familiar with we have used for a very long time.

So it is just real numbers but also the notion of i, the imaginary unit is also quite familiar where if there is no real part. But there is only an imaginary part and that imaginary part with magnitude 1 right. So, it is customary to call x as the real part of z and y as imaginary part of z if denoted as Re(z) and Im(z).

So, the addition of 2 complex numbers is defined as simply adding the real parts and the imaginary parts, 2 complex numbers each of them contains these 2 you know pieces of information and the different pieces of information add separately.

So, z is equal to z 1 plus z 2 means that its real part is going to be x 1 plus x 2 and imaginary part is going to y 1 plus y 2. And so multiplication of 2 complex numbers is defined by this operation. So, you take the the real part of the first complex number and multiply it by the real part of the second complex number and then subtract the product of the imaginary parts of the 2 complex numbers.

So, if you are trying to multiply z 1 times z 2. So, the real part of the resulting complex number right. So, we just have to specify the real part and the imaginary part of the resulting complex number and so that is going to the real part is going to be given by just you know the product of the real part of the 2 complex numbers minus the product of the imaginary part of the 2 complex numbers which are being multiplied.

And so the imaginary part of the resulting complex number is given by the sum of the products of the real and imaginary parts. So, you have x 1 times y 2 plus y, y 1 times x 2 right. So, this is also something we have used, but the thing is that starting from this definition we can actually derive the identity:

$$i^2 = -1$$

We have probably encountered i as a solution to the algebraic equation $z^2 + 1 = 0$ and then we just simply introduce a label called i and so that is supposed to be the root of this algebraic equation for which there is no solution in real numbers.

And so of course we know that i squared equal to minus 1 because that is often the starting point where we have encountered this. But you can actually derive this thinking of it in an axiomatic way like here. So, if you define multiplication of 2 complex numbers like here and the imaginary i is defined as 0 comma 1 right.

So, i squared is nothing but the product of these two complex numbers 0 comma 1 times 0 comma 1 and then we have this definition of product of 2 complex numbers. So, you have to do 0 times 0 minus 1 time is 1 and 1 time comma 1 comma 1 times 0 plus 0 times 1 which is just minus 1 comma 0 and often if it is just a real number you omit writing the 0 and you just identify it as just minus 1 that is a real number right.

So, it is possible to build this framework up in an axiomatic way. We will not push this too far, but I am just recalling a set of definitions and we will state certain properties and use these properties and get going in an intuitive way right. So, it is possible to spend a lot of time and do it in a very thorough rigorous way and there are math text books available where you can look up this way of building the whole structure.

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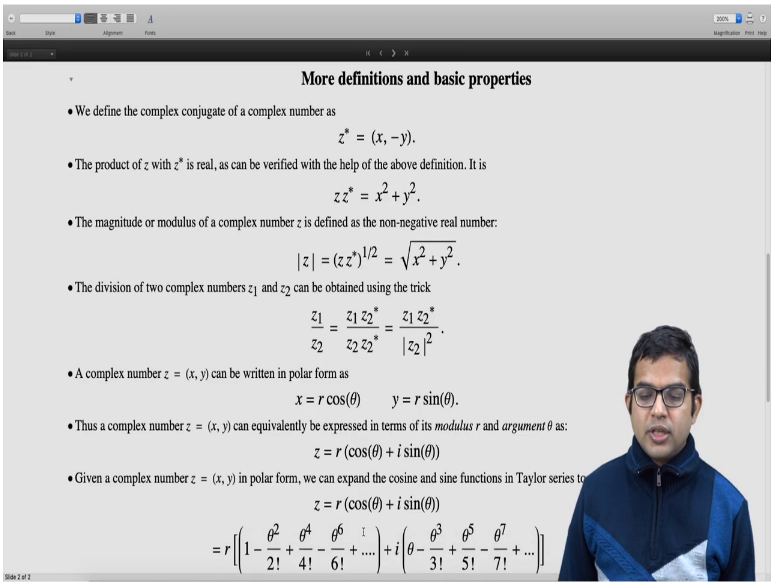

Some more definitions and some basic properties again stuff which we should be familiar with. So, the complex conjugate of a complex number is another complex number, which also has a real part and an imaginary part you just have to specify these two to specify a complex number. And here the real part of this complex conjugate remains unchanged and the imaginary part has a change of sign so x comma minus y right.

So, z star is another complex number and so we will see how the complex conjugate makes a reappearance later on when we are studying the theory of analytic functions and so on. But even otherwise so just as you know when simple operations involving complex numbers are involved. So, this has great utility. So, there is the notion of a complex conjugate of a complex number.

So, the product of z with z star is real as can be verified from first principles if you want invoking the definition of multiplication and so you can check that. In fact, z times z star is nothing but x squared plus y squared or the sum of the squares of the imaginary and the real part of the complex number. So and of course the complex conjugate of the complex conjugate of a complex number is itself.

So, because minus y you get a minus 2 times and then you go return to the original complex number itself. So, the magnitude or the modulus of a complex number can be defined using the complex conjugate. So, it is written like here mod of z is just the square root of z times z star.

So, we are guaranteed that the product of a complex number with its complex conjugate is not only real, but it is also positive because you have the sum of 2 positive numbers x squared plus y squared. So, the square root of this quantity is definitely you know well defined it is a real number and it is the positive square root that you are looking at.

It is a non-negative quantity a real quantity mod of z right and it is it is the magnitude of a complex number z, which is defined as the square root of this sum of the squares of the real and imaginary parts of the complex number. So, the division of 2 complex numbers is something which you know basically follows from the definition of multiplication.

So, z 1 time divided by z 2 you know can be written as z 1 times z 2 star divided by you know you just multiply and divide throughout by z 2 star and then make use of the fact that the denominator now becomes a real number. So, then it is just like a multiplication of 2 complex numbers z 1 times z 2 star right. So, this you know this property and similar properties where you can you know you can there are tricks associated with the modulus and it has great utility.

Now, a complex number can also be written in polar form right. So, the x the real part can be thought of as r times cosine of theta and the imaginary part can be thought of as r times sine theta; where this r is actually nothing but the modulus of the complex number.

And so there is a modulus of the complex number and then there is theta is to the argument of a complex number. So, there is this geometric representation possible for a complex number you can think of it as essentially a vector in the 2 dimensional a plane and then so theta is the amount by which your vector has rotated with respect to the x axis.

So, the x axis is nothing but the real axis and the y axis plays the role of the imaginary axis. And so this modulus of z is the same as this the radius of this you know the vector corresponding to the complex number that you are thinking about. And so you can basically write down your complex number z as r times cos theta plus i times r sin theta, so because the real part is just r cos theta and the imaginary part is r sin theta.

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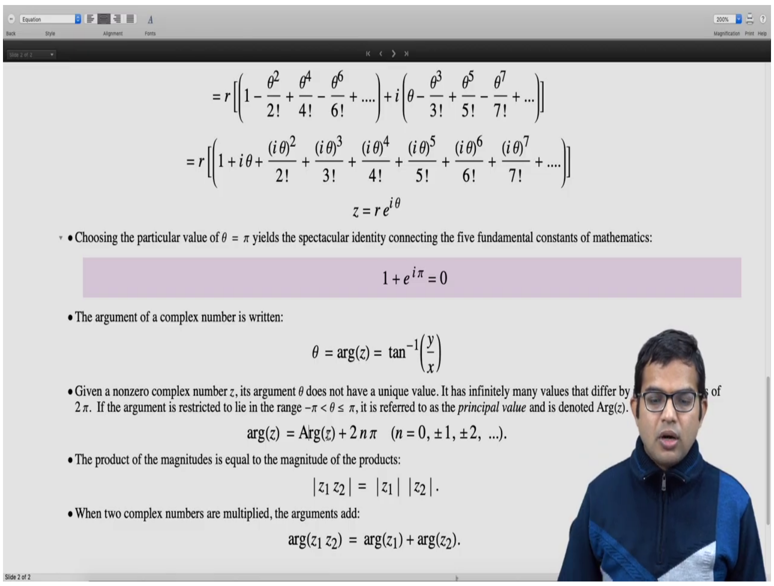

So, given a complex number in polar form we can go ahead and expand this the cosine and sine functions in a Taylor series right. So, each of them separately in two Taylor series and so cosine theta has this expansion 1 minus theta squared by 2 factorial plus theta into the 4 by 4 factorial minus and so on.

And likewise sine theta has this Taylor series expansion theta minus theta cube over 3 factorial plus theta to the 5 over 5 factorial so on right. You know carefully collect these terms together and then you have a 1 and then plus i theta and then minus theta squared by 2 factorial can be written as i theta the whole squared divided by 2 factorial.

Because we know that i squared is minus 1 and likewise you know we can write down minus theta cube by 3 factorial times i can be written as i theta the whole cube divided by 3 factorial, because i squared will give you minus 1 and i remains as it is. So, this minus sign is taken care of then we write down the fourth term as i i times theta to the whole power 4 divided by 4 factorial, because i to the power of 4 is just 1.

And so likewise so you can check that indeed all of these terms become positive if we bring in this i and place it you know in this strategic manner and then we collect terms together to write down this one common series. And then we see that these series are nothing but the series expansion for the exponential.

So in fact we identify that z is equal to r times e to the i theta. So, this function cos theta plus i sin theta is actually nothing but e to the i theta. So, this is not a completely rigorous derivation. But this is an argument if you wish and so this goes back to Euler, so this identification of this exponential i theta as cosine theta plus i sine theta. So, the real part of e to the i theta is this cos theta and the imaginary part of e to the i theta is sin theta.

And so this equation is extremely convenient. It allows us to prove all kinds of remarkable results and so one particular very beautiful result which immediately follows from this is when you put theta equal to pi. If you put theta equal to pi we have

$$e^{i\pi} + 1 = 0.$$

So, we have cos pi is minus 1, so minus 1 you bring to the other side and then i times sin pi is 0 so indeed. So now, you see that this equation has 1 it has e it has pi it has i and it has 0 these are all the you know fundamental constants of mathematics, all 5 of them together appear in this very beautiful equation. I guess this equation also goes back to Euler; it is one of those remarkable and celebrated equations.

So, the argument of a complex number is this thing called theta and there is this notation for the argument of a complex number and it is just written as argument of z and it can be thought of as tan inverse of y over x right. So, because the real part is cos theta and the imaginary part is sin theta I mean there is an overall magnitude as well and so this will cancel out and so it is just going to be tan inverse of y by x right.

So, you can write an expression for cos theta and for sin theta and then divide y by x and then show that indeed tan theta is just 1 by x. So therefore, you get this expression and so the key point to note here is that this theta is not unique, it has in fact infinitely many different values and they all differ by integer multiples of 2 pi. So, tan inverse is you know you know this function arg of z is sometimes also called a multi-valued function.

So, there is a way to make this get a unique value and that is to restrict your answer to lie within a range and so the standard range that is taken is minus pi to plus pi. So, if you restrict the argument to lie within this range minus pi to plus pi, then the function is denoted with the capital A.

So, $arg(z) \ = \ Arg(z) \ + \ 2\,n\,\pi$

So, you know arg of z is there is a there is one value for capital Arg of z, but there are always infinitely many different values that are associated with small arg right. So, this notion of non uniqueness is something which has consequences and then perhaps much later we might return to this you know this aspect of complex functions which do not have a unique value.

Now, the product of the magnitudes of 2 complex numbers is simply the is the magnitude of the products very straightforward, mod of the z 1 times z 2 is just mod of z 1 times mod of z 2 and when 2 complex numbers are multiplied their arguments add. So, this is the small arg you should be careful you know capital Arg may not add, because this restriction sometimes it does not quite add.

Although you know if z 1 and z 2 are of a certain special kind or there are if the arguments are in certain values this equation may hold or but in general capital Arg of z 1 times z 2 is not equal to capital Arg of z 1 plus capital Arg of z 2. But this equation does indeed hold if small arg is involved

I mean we should always you know allow for the fact that each of these args is comes with a you know is modulo some free multiple of 2 pi right. So, within that extra freedom this equation certainly holds. So, the small arg is there is this there is an extra freedom which can always be exploited and so this equation does indeed hold ok.

So, most of the material that we have covered in this lecture are probably already known to all of us but. So therefore, this must be thought of as in the nature of recall and we will start using some of these properties starting from the next lecture.

Thank you.