Mathematical Methods 2
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Complex Variables
Lecture - 11
More persepective on differentiability

So, we have seen how differentiability implies Cauchy-Riemann conditions and Cauchy-Riemann conditions are also part of the sufficiency requirements. Now, there is a way to think about these Cauchy-Riemann conditions in a much more direct way. So, a function of a complex variable is differentiable only if the function is of a very special kind right and that is something that we can sort of directly see from the function. And so that is the perspective which we want to provide in this lecture right.

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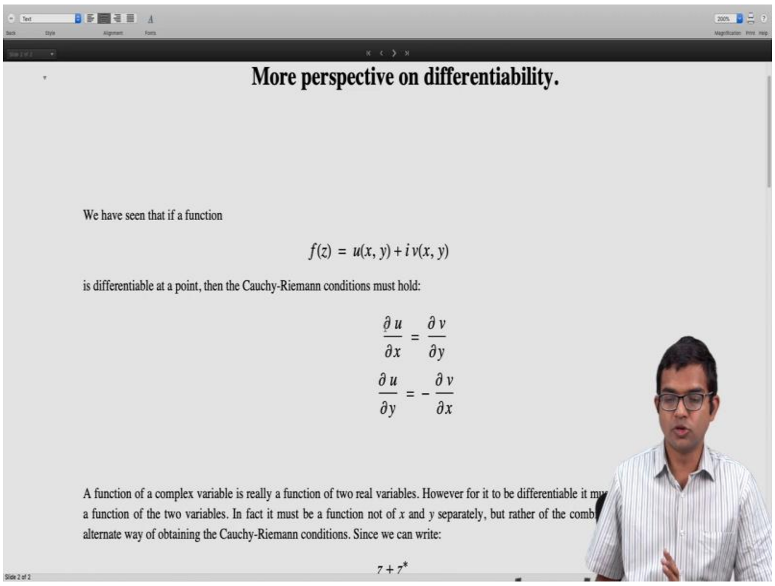

So, once again, so we see that if your function f of z is equal to u of x,y plus i times v of x,y, if these conditions hold dou u by dou x to dou v by dou y and dou u by dou y equal to minus dou v by dou x, then Cauchy-Riemann conditions hold and these functions are smooth enough defining well neighborhood of this point. Then, the function is differentiable.

So, the thing is that a function of two variables any arbitrary function is not going to make your overall function of the complex variable differentiable right. So, we might think of this as really being made up of two variables x and y and so, you tie together any two functions u and v, that is not going to make a nice enough function for f right. So, I mean although a complex variable has two variables and you are thinking of a function of two variables, a function of a complex variable is slightly more than just this.

So, in fact, it must be a function of this very nice combination of these two variables. Now, any arbitrary combination does not work. So, it must be of this kind, z its a function of z right. So, we write it as f of z and so, only whenever x appears, x plus iy must appear in that way; only then will your function be you know a nice function and that is basically the content of the Cauchy-Riemann condition. So, this is something that we can see more directly and that is what we are going to do here.

So, since x is the real part of z. So, we can write x as z plus z star over 2 and y is the imaginary part. So, we can write it as z minus z star over 2 i right.

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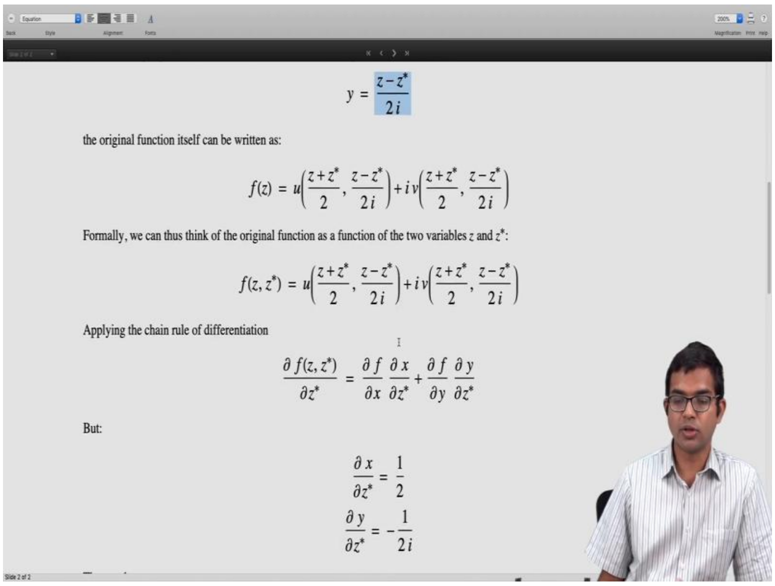

So, the original function itself can be thought of as follows. It has its u of x,y plus i times v of x, y, but you can write it as u of z plus z star over 2 comma z minus z star over 2 i plus i times v of z plus z star over 2 comma z minus z star over 2 i. So, you can actually think of this function as being made up of these two independent variables. So, think of z star itself as an independent variable and let us see what is what happens if you take this derivative.

Suppose, we take this partial derivative with respect to z star right. So, dou f by dou z star is the same as dou f by dou x and so, we are applying the chain rule. So, dou f by dou x. So, you think of this function as having two independent variables; z, z star. So, to make that explicit, I have written f as like here and although, we are used to think of f as just as a function of z.

But you know z itself is made up of two independent variables, x, y or we could also think of this function in general as being a function of you know these two independent variables, z, zstar. So, if you take this partial derivative to f with respect to f with respect to z star to f by dou z star, then you can write it as dou f by dou x by dou x dou z star plus dou f by dou y times dou y by dou z star.

But dou f by; so, dou x by dou z star is nothing but half from here we know this and dou y by dou z star is nothing but minus 1 by 2 i right. So, we have these two relations which come from the definition right.

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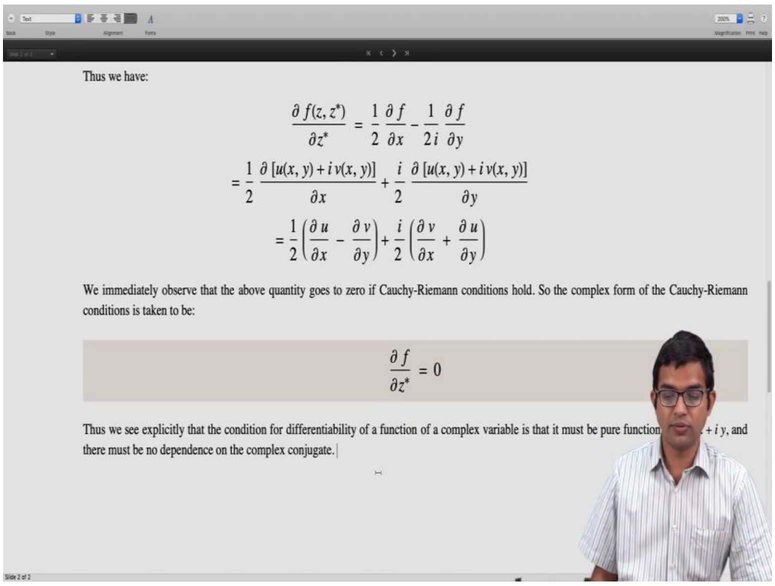

So, dou f by dou z star is nothing but half dou f by dou x minus 1 over 2 i dou f by dou y and we can write f you know expand it out and write it as u of x comma y plus i times v of x comma y and then, we can collect all the terms which are you know the real part and the imaginary part separately. So, we see that this is nothing but half dou u by dou x minus dou v by dou y plus i by 2 times dou v by dou x plus dou u by dou y right.

So, we immediately observe that this is in a very suggestive form right. So, this combination, we are very familiar with and so, are we with this combination. So, this quantity is dou f by dou z star is going to vanish with the Cauchy-Riemann conditions hold. So, dou u by dou x is equal to v by dou y and dou v by dou x is equal to minus dou u by dou y. So, in fact, this condition dou f by dou z star equal to 0 is equivalent to the Cauchy-Riemann condition.

So, this is what is called a complex form of the Cauchy-Riemann conditions. And so, if you pause for a moment to think what this means, so its the statement that this function f should be dependent only on z and it should have no dependence on z star right. So, explicitly, so with the that is what is basically the content of the Cauchy-Riemann conditions right.

So, then and it's only when that happens that another derivative of a function f with respect to z is well-defined, only if it has no dependence on z star right. So, that is just another way of thinking about Cauchy-Riemann condition, which we worked out first of all looking at properties in the Cartesian system, but also Cauchy-Riemann conditions in the polar coordinate system as well. But essentially, it is just the statement that there is no role for z star right.

So, now in the backdrop of this understanding, if we go back to some of our examples, we see whenever we had z star, we ran into trouble. Such functions were not nice functions for differentiability. And now, there is you know an understanding for why that was happening comes on, if we think about you know f as a function of z , z star and if only functions which are purely functions of z are the ones which are differentiable.

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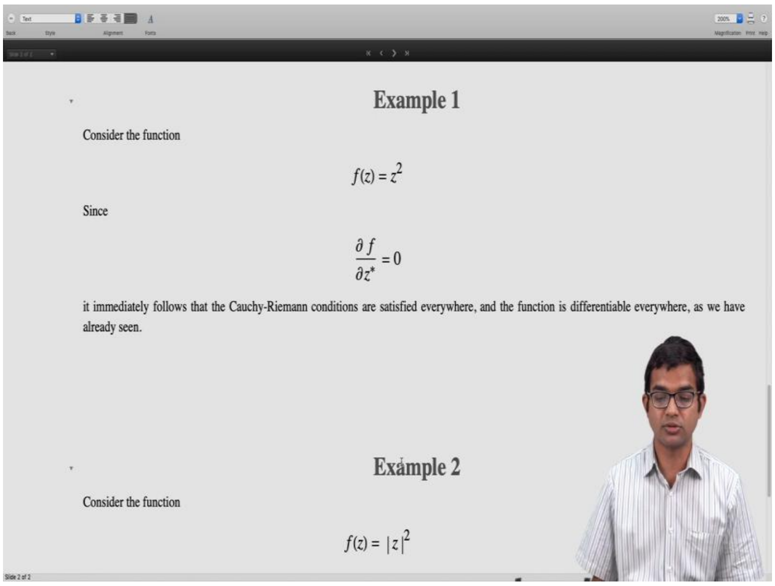

So, let us go back and look at two examples which you have already seen. So, f of z equals z star and z squared is a very nice function, we never had any difficulties here and we see immediately that dou f by dou z star is equal to 0. There is no dependence on z star. It is purely a function of z alone. Therefore, we can go ahead and take a derivative. We know that its derivative is just to z everywhere. So, its Cauchy-Riemann condition will hold you know everywhere. So, it is a very nice function.

On the other hand, if you look at this function f of z is equal to mod z squared. So, this is where we ran into difficulties. Whenever you have something to do with z star and so, there is no way to represent the information in mod z squared without recourse to z star.

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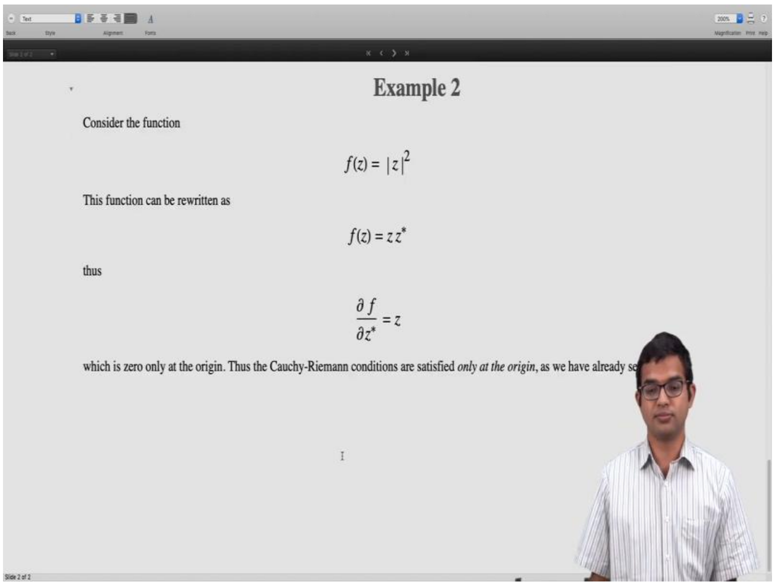

So in fact, this function is nothing but z times z star right. So, z cannot give you what z star does. Well, there is no way to get all the information in this without using z star.

Therefore, if you do dou f by dou z star that is going to be z and this we have seen is 0 only; mean this is 0 at only at the origin and so, indeed the Cauchy-Riemann conditions are satisfied only at the origin as we have already seen and this particular function, it turns out is differentiable at the origin right and Cauchy-Riemann conditions hold, only at the origin and nowhere else right.

So, this is something we already see, but this is some extra perspective for the Cauchy-Riemann condition, ok.

Thank you.