Mathematical Methods 2
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Complex Variables
Lecture - 12
The Value of the Derivative

So we have seen how Cauchy-Riemann conditions hold whenever the derivative is well defined for a function of a complex variable. You have also seen that the Cauchy-Riemann conditions and some continuity properties; together are also sufficient conditions for a derivative to be meaningful.

But we did not explicitly point out what the value of the derivative of a function would be when it is well defined. So, we will quickly use this lecture to explicitly write down the value of the derivative when it exists ok.

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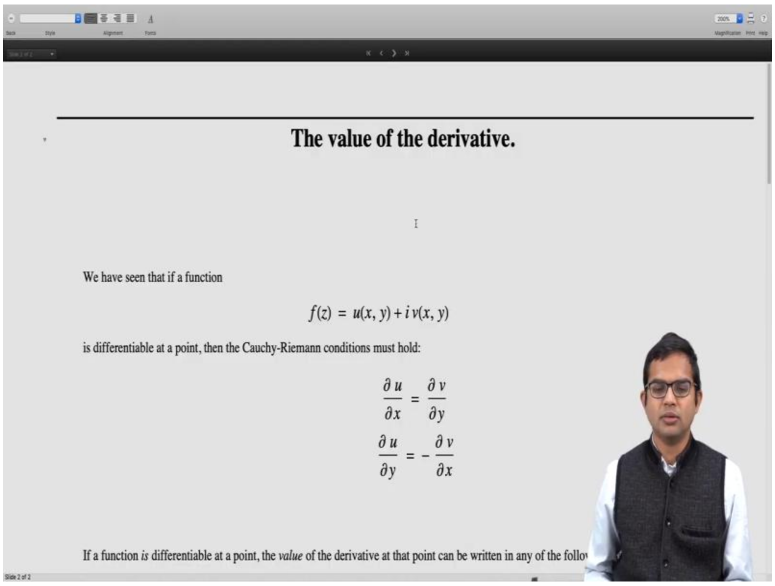

So, you have given some function f of z at (Refer Time: 01:08) as is customary we write it as u of x,y plus i times v of x,y. And if it is differentiable at a point then the Cauchy-Riemann conditions hold: dou u by dou x is equal to dou v by dou y and dou u by dou y is equal to minus dou v by dou x. So, the value of the derivative is also something that can be represented in terms of these partial derivatives right.

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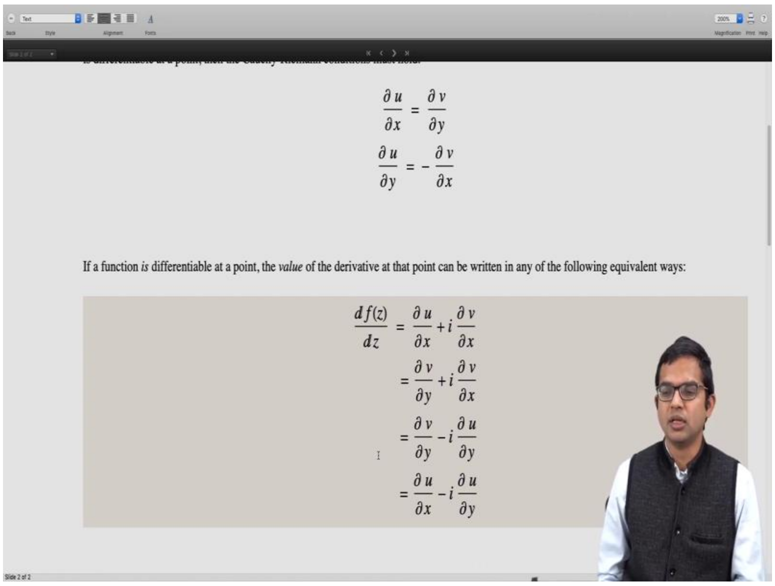

So, any of these following four equivalent ways are acceptable right. So, in fact you know one way in which we argued that you know Cauchy-Riemann conditions well must hold is by trying to work out this derivative in different directions right. So, if you recall from one of the earliest discussions around Cauchy-Riemann conditions. So we in fact, argued that if you have approaching one direction you would get to dou u by dou x plus i times dou v by dou x.

But I mean you can also derive one of these and then simply use the Cauchy-Riemann condition, so you have dou u by dou x is equal to dou v by dou y. So, in place of dou u by dou x you can put dou v by dou y and leave dou v by dou x as it is, or it may be sometimes more convenient to you know replace both of these.

So, you have dou v by dou y for the real part and you write the imaginary part as minus dou u by dou y. Or it may be sometimes convenient to just work out dou u by dou x and then the imaginary part to be just minus dou u by dou y correct. So, all of these are really the same and they better be right so that is the content of the Cauchy-Riemann conditions.

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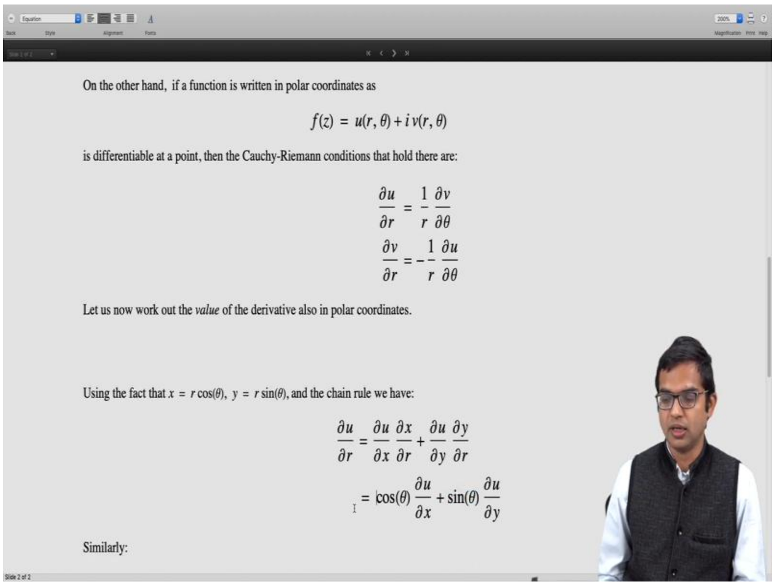

So, if a function is differentiable the value of the derivative can be evaluated in any of these above ways also if a function is written in polar coordinates right. So, then also it is convenient to write the real part and imaginary part of your function, but each of these real part and imaginary part must be thought of as functions of r and theta rather than of x and y right. So, Cauchy-Riemann conditions we saw where dou u by dou r is equal to 1 over r dou v by dou theta of course, we assume that r is not 0.

So, dou v by dou r is equal to minus 1 over r dou u by dou theta. Now, the value of the derivative can also be worked out in polar coordinates. So, using the fact that you know x and y are defined as r cos theta and r sin theta and using the chain rule. So, we have dou u by dou r is equal to dou u by dou x times dou x by dou r plus dou u by dou y hence dou y by dou r.

But dou u by dou x you leave it as it is and dou x by dou r is the same as cos theta and dou y by dou r is sin theta.

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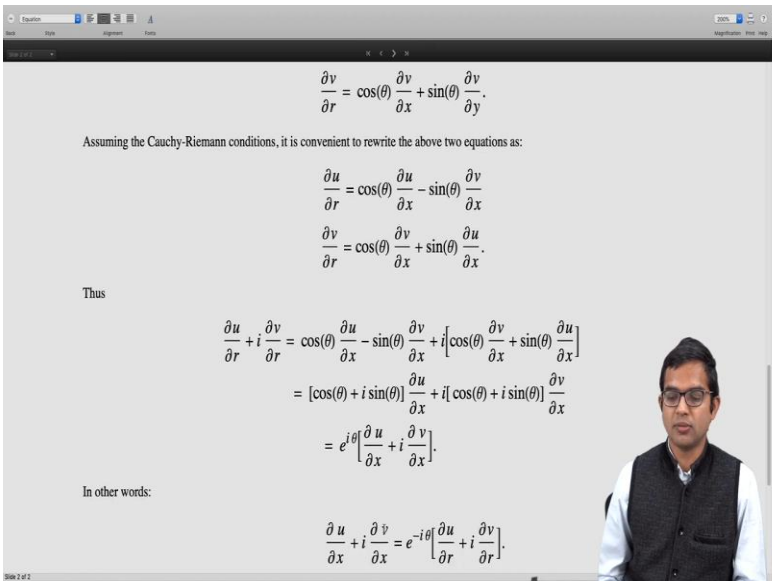

So, this dou u by dou r is the same as cos theta dou u by dou x plus sin theta dou u by dou y. So, similarly if you took this other function dou v by dou r you know take the function v of r comma theta and find the partial derivative with respect to r. So, again you will get the same kind of a relation dou cos theta dou v by dou x plus sin theta dou v by dou y.

And then using Cauchy Riemann conditions it is convenient to rewrite these two equations. So, dou u by dou r you know write is equal to cos theta dou u by dou x you write it as it is. But in place of dou u by dou y its convenient to write minus sin theta dou v by dou x right so we will see in a moment why this is.

So, and likewise dou v by dou r we write cos theta dou v by dou x as it is and then in place of dou v by dou y you just write down dou u by dou x right. So, that is the Cauchy- Riemann conditions we have applied you know here and here you know in place of dou u by dou y and in place of dou v by dou y, we have written minus dou v by dou x and dou u by dou x right.

So, once we have these two equations we just add the two, but with a factor of i associated with the second of these equations. So, it should be dou u by dou r plus i times dou v by dou r when you get cos theta dou u by dou x minus sin theta dou v by dou x plus i times cos theta dou v by dou x plus sin theta dou u by dou x. Now, rearranging your cos theta plus i times sin theta you know is multiplied by dou u by dou x.

Then if you pull out an i then you have a cos theta you know this is minus sin theta can be written as plus i squared sin theta and we plot one of these i s. So, you have cos theta plus i sin theta you know both of them have this factor dou v by dou x which comes out and then we see that this factor cos theta plus i sin theta is common. So, you pull it out and also we use this Euler identity. So, cos theta plus i sin theta is the same as e to the i theta and we have this expression dou u by dou x plus i times dou v by dou x.

So, in other words we have managed to show that dou u by dou r plus sorry, dou u by dou x I mean I write it write you know this right hand side. I have pulled out only this dou u by dou x plus i times dou v by dou x is the same as e to the minus i theta dou u by dou r plus i times d v by d r dou u dou v by dou r right. So, but what is this quantity? This quantity is nothing but the value of the derivative at that point in Cartesian coordinates.

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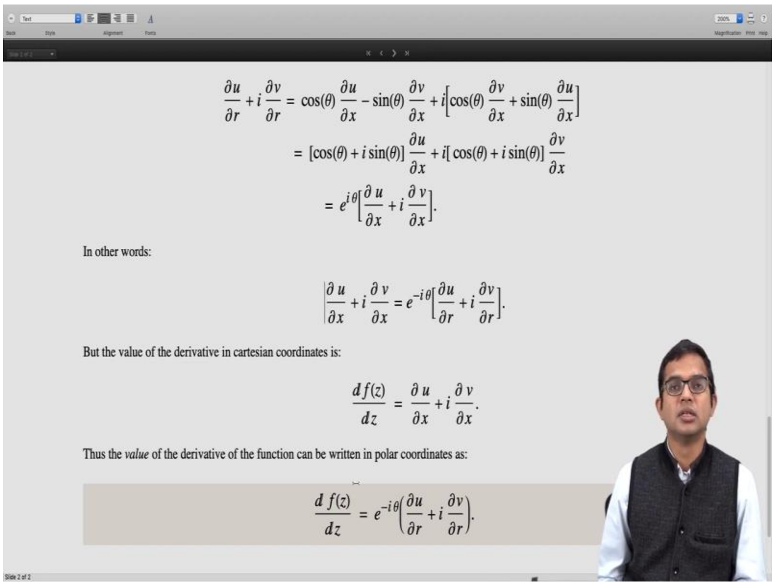

So, the value of the derivative in Cartesian coordinates is this d f by d z is equal to dou u by dou x plus i dou v by dou x. Therefore, the value of the derivative of the function in polar coordinates can be written as simply e to the minus i theta times dou u by dou r plus i times dou v by dou r right.

So, often it is convenient to work out the value of the derivative directly in polar coordinates right. So, sometimes it is much more tedious to do the check for Cauchy- Riemann conditions in polar in Cartesian coordinates and therefore, it is quite useful to have this polar form ready.

And so, that is what we did in this lecture. We worked out the value of the derivative both in Cartesian coordinates and in polar coordinates.

Thank you.