Mathematical Methods 2

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Module - 02
Complex Variables
Lecture - 14
Harmonic functions

So, we have defined analyticity, we have looked at some properties of analytic functions, and we looked at a few examples. In this lecture, we look at what are called harmonic functions, and how the idea comes about naturally when we are looking at an analytic function.

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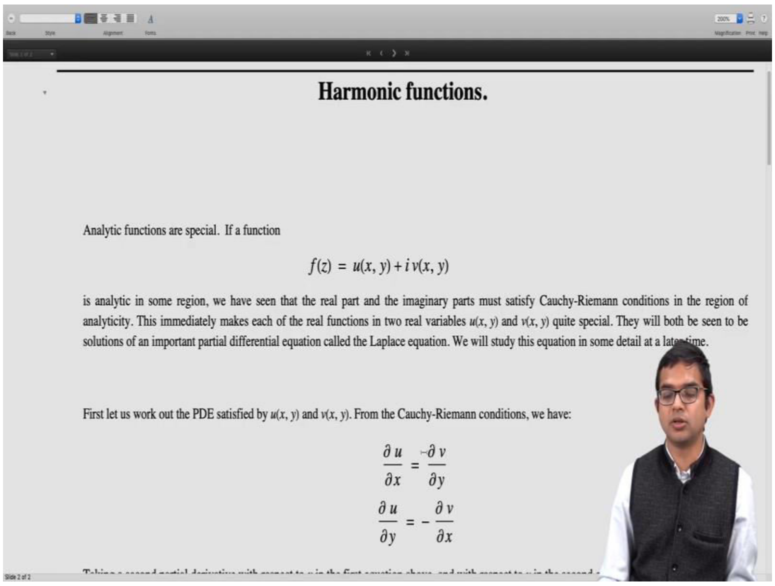

Yeah. So, analytic functions are special right. So, they satisfy Cauchy-Riemann conditions, their real part and imaginary parts do, and they do so in a region. So, this gives them some very nice properties right. So, in this lecture, we will see how the real part and the imaginary part, because of their satisfying Cauchy-Riemann conditions, also individually satisfy a partial differential equation which is called the Laplace equation.

And it is something that we will return to later on in this course as well like when we are studying partial differential equations. But here we point out that an analytic function naturally gives rise to functions which satisfy the Laplace equation. So, let us start with the

Cauchy-Riemann conditions. We are given u and v and so we know that dou u by dou x must be equal to dou v by dou y, and dou u by dou y must be equal to minus dou v by dou x.

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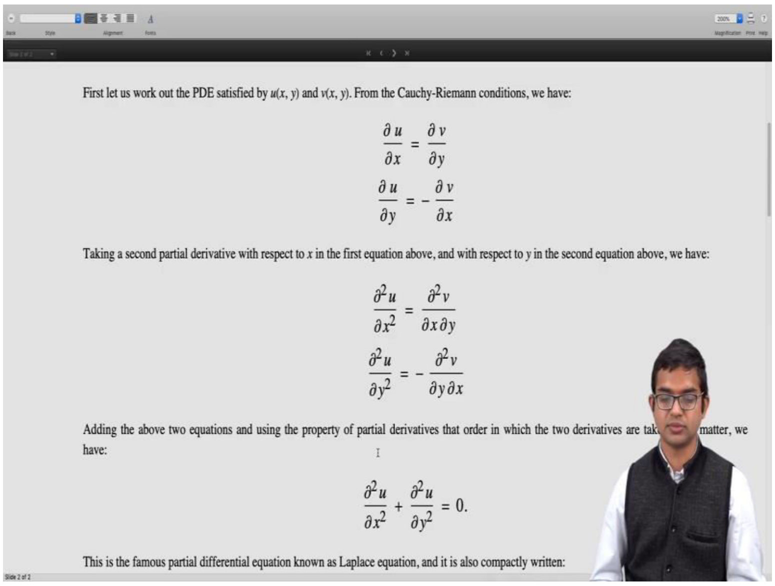

So, if you take a second partial derivative with respect to x in the first equation and with respect to y in the second equation, we have dou square u by dou x square is equal to dou square v by dou x dou y, and dou square y by dou y square is equal to minus dou squared v by dou y dou x right.

So, we see that the right hand side in these two equations are really the same except for a minus sign right, because it does not matter in which order you take you know this partial derivative. So, dou square d by dou x dou y is the same as dou squared d by dou y dou x. So, when you add these two, you get dou square u by dou x squared plus dou square u by dou y squared equal to 0. So, this is the famous Laplace equation right.

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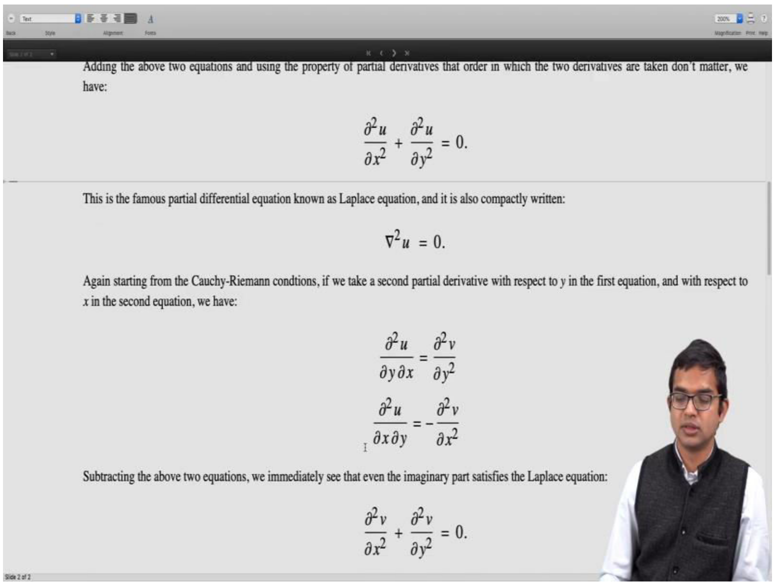

So, it is also written as nabla square del squared u is equal to 0. So, it is something that we have perhaps encountered in electrostatics or when we are looking at properties of temperature distributions and so on right. So, if we started with the Cauchy-Riemann conditions we could have also taken the second partial derivative in a slightly different way.

So, suppose we took a second partial derivative with respect to y in the first equation and with respect to x in the second equation, so then we would have you know dou square u by dou y dou x is equal to dou square v by dou y square, and dou square u by dou x dou y is equal to minus dou square v by dou x square. If we subtract these two equations, then once again we will see that in fact even the imaginary part also satisfies the Laplace equation dou square v by dou x square plus dou square v by dou y square also is 0 right.

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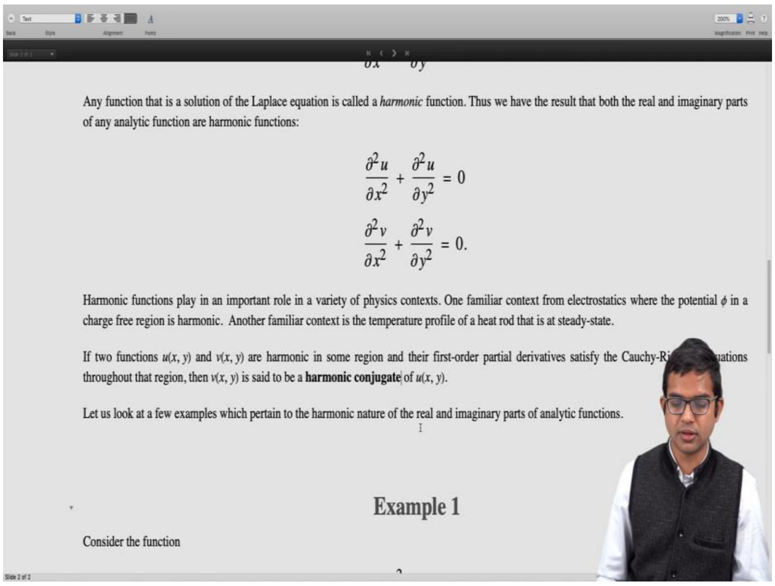

So, any function which satisfies the Laplace equation is called a harmonic function right. So, we have the result that an analytic function automatically gives us two harmonic functions – the real part of an analytic function and the imaginary part of an analytic function, both are harmonic functions.

But it is not enough for us to take together any two harmonic functions and then couple them together with an i and then you cannot have an analytic function. Analytic function is more than just two harmonic functions right. So, an analytic function for sure gives you two functions which are harmonic, but it is a little more than just that right.

So, harmonic functions like we said earlier - we have seen them in many physical contexts and they have importance right. One is in electrostatics, when we are looking at the potential phi in a charge free region. Another is the context of the temperature profile of a heat rod which is in steady state right.

So, we will return to partial differential equations, we will think about partial differential equations and how to solve them and all this you know a little bit later in this course. And then we will discuss methods to solve such differential equations and so on.

But let us look at Cauchy-Riemann conditions and how harmonic functions come from analytic functions here. So, there is one more definition which we want to give here, which is that if you have two functions which are harmonic in some region.

And in addition to this, the first order partial derivatives satisfy the Cauchy-Riemann conditions throughout that region, so in other words, if you got your u and v from an analytic function, then you say that v of x, y is said to be a harmonic conjugate of u of x comma y right. So, yeah, so u and v not only satisfy the Laplace equation individually, but there is also a relationship between the two which is important if they have to couple together to give you an analytic function.

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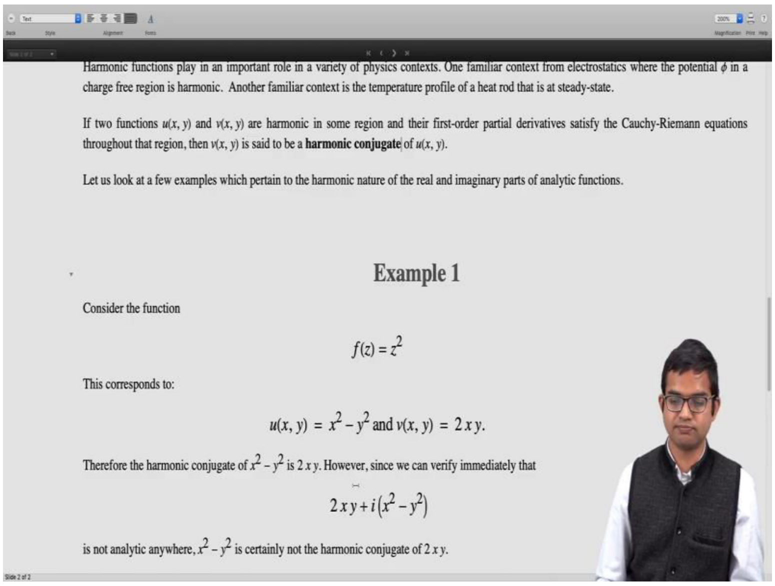

So, let us look at a few examples which pertain to this harmonic nature. So, suppose we consider our favorite function f of z is equal to z squared. So, this corresponds to u of x comma y which is x squared minus y squared, and v of x comma y is equal to 2 x y. Then we can immediately say that you know the harmonic conjugate of x squared minus y squared is 2 x y right.

Of course, we can first of all make the statement that each of these is a harmonic function not only are they both individually harmonic in nature, but v is the harmonic conjugate of u right. So, this order is also important. You cannot say that u is the harmonic conjugate of b. So, in fact, it is not true because you know the function 2 x y plus i times x squared minus y squared is not going to be analytic.

You can check this by working out the Cauchy-Riemann conditions right. So, the order in which you put these terms is important. So, x squared minus y squared is certainly not the harmonic conjugate of 2 x y, but the harmonic conjugate of x squared minus y squared is indeed 2 x y everywhere right. So, this is an entire function.

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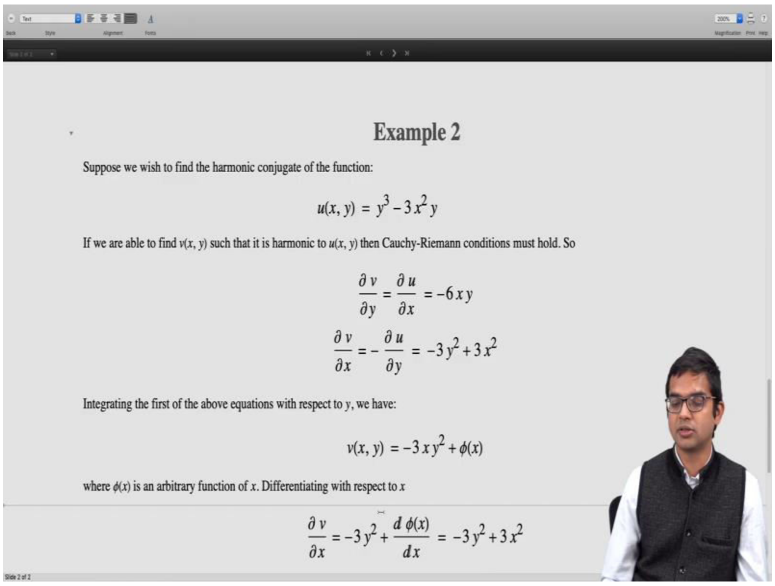

Let us look at another example. So, suppose we are given some function u of x comma y right, y cube minus 3 x squared y. And we ask if it is possible to find its harmonic conjugate. In other words, if the real part of an analytic function is given to us, can we find a suitable imaginary part such that we can tag them together and make an analytic function?

So, this may not in general be possible. You may have functions you know real functions of two variables for which you know which cannot just be made the real part of an analytic function right. It may be impossible to find a harmonic conjugate for a function.

But here it is going to be possible. So, let us see the technique for how to find the harmonic conjugate of this function. So, what we need is to find a v such that Cauchy-Riemann conditions must hold. So, dou v by dou y must be equal to dou u by dou x, but dou u by dou x is something you can work out here, dou u by dou x is simply minus 6 x y. And we also want dou v by dou x to be equal to minus dou u by dou y.

And again minus dou u by dou y we can work out, so it is just minus 3 y squared plus 3 x square right. So, we know what dou v by dou y is which is minus 6 x y and we also know what dou v by dou x is which must be equal to minus 3 y square plus 3 x square. And this is enough to work out v up to a constant.

So, integrating the first of these, so immediately we have v of x comma y must be minus 3 x square xy squared plus some arbitrary function of x So, we have this freedom right because dou v by dou y regardless of what phi is going to be minus 6 xy as we can check by taking the partial derivative with respect to x.

Again, if yeah, so that is exactly what we do next. If you take a partial derivative with respect to y, phi of x does not matter right. So, because dou phi by dou y is definitely 0; as far as y is concerned, phi of x is a constant. But on the other hand now if you differentiate this with respect to x.

So then we have dou v by dou x is equal to minus 3 y square plus now you do not need a partial derivative with respect to x, it is just a full derivative d by d x of phi of x which we already know dou v by dou x must be equal to minus 3 y squared plus 3 x squared. So, this 3 minus 3 y squared will cancel on both sides.

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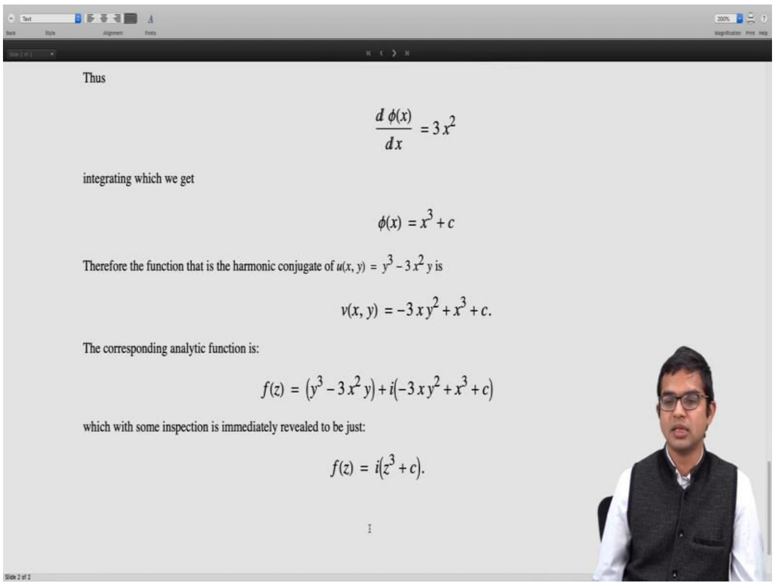

And we are left with d phi by dx is equal to 3 x square which is something which is very straightforward we can immediately solve it. And we have phi of x must be equal to x cube plus some arbitrary constant. So, therefore, the function that is harmonic conjugate of u of x comma y is simply minus 3 xy squared plus x cube plus some constant.

So, in fact, you can show that given a function if it has a harmonic conjugate, it is unique up to a constant right. So, you can add some arbitrary constant, but not much else right. So, in this case, the corresponding analytic function itself turns out to be quite easy to read off, and write it in a very compact form.

So, let us write it down. So, f of z is y cube minus 3 x squared y plus i times minus 3 x square x y squared plus x cube by plus c. And some inspection reveals that in fact this function f of z is just i times z cube plus c right. So, indeed, it is an analytic function and it is just a polynomial.

So, basically there is no dependence on z star. So, indeed the partial derivative of f with respect to z star is going to be 0. So, this is an example where you know we managed to find the harmonic conjugate of a function because the harmonic conjugate exists. There are other examples where it is not possible to find a harmonic conjugate.

So, we could play this kind of game. Take an arbitrary function u of x comma y and try to find its harmonic conjugate right. So, it should be quite straightforward to find such a function right. So, yeah so that would happen if you know it is not possible to tack together u and v, and together make it into an overall analytic function.

So, the main message from this lecture is that whenever you have an analytic function it gives you a real part and imaginary part, which are necessarily harmonic functions, but they are also something slightly more than that. So, the Cauchy-Riemann conditions also connect them in some very special way. So, that is what this lecture is about.

Thank you.