Mathematical Methods 2
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Module - 02
Complex Variables
Lecture - 16
Complex Logarithm

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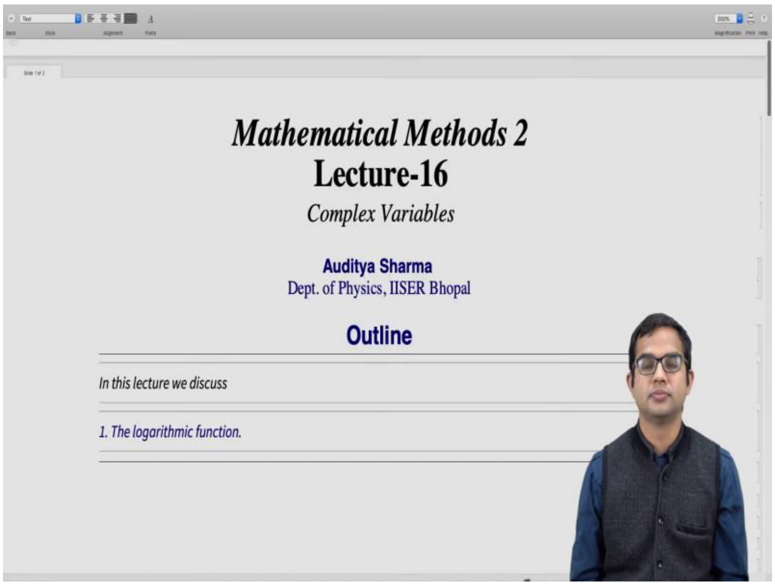

Ok so, we have seen how a generalization of the exponential function to include complex variables, possible in such a way as to leave it analytic and it turns out that the exponential function so defined is actually analytic in the entire finite complex plane and it is an entire function right. So, we also said that for any complex number in the complex plane, it is possible to work out the inverse right. So, it is possible to find the corresponding complex number, whose exponential will be a given complex number, if we go in that direction.

So, we encounter the logarithmic function, which is the subject matter for this lecture.

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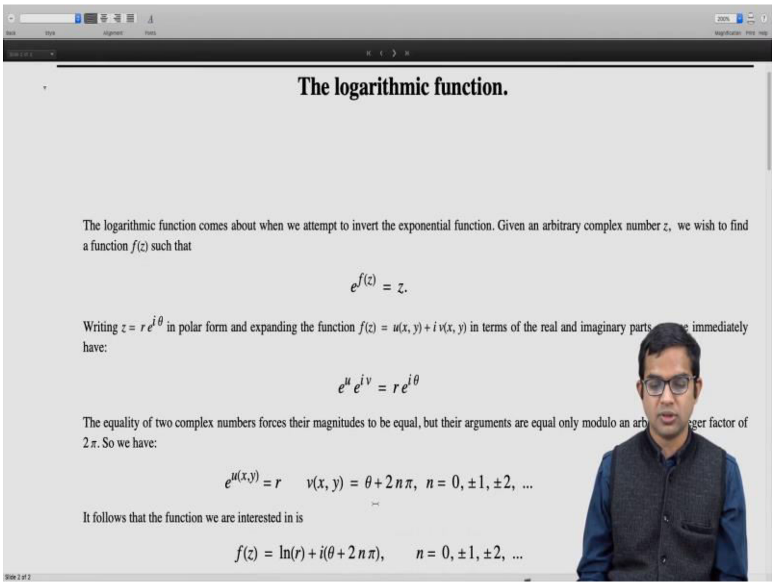

So, given an arbitrary complex number z, we wish to find a function f of z such that, the exponential of this function e or the value from this function e to the f of z must give back z right. So, if we write down z in polar form as r times e to the i theta, and expand the function f of z itself as a real part and complex parts. So, u of x comma y plus i times v of x comma y and. So, you immediately see that this is the same as asking e to the u times e to the i v right.

So, this is a property of exponential over the complex number, it is like this if you are going to take the exponential of the sum of two complex numbers, it is the product of the exponentials of the two complex numbers. So, you have e to the u times e to the i v must be equal to r times e to the i theta.

So, now, you see that this equation on the left hand side here, you have e to the u which is exponential of a real number. So, it must be you know it is the usual exponential function. So, it is. So, there is a real part and then e to the i v is where you get the complex part at all.

So, it is a phase which also appears on the right hand side. So, whenever you are comparing two complex numbers and if you want their equality, we know that the modulus of each of these complex numbers must necessarily be the same. So, e to the u which is the modulus, which appears on the left hand side must be equal to r, and also if e to the i v is equal to e to the i theta.

So, we are saying that the arguments of this complex number are the same, there is some freedom right.

So, as far as the relationship between v and theta is concerned they do not have to be exactly the same, but v of x comma y, can differ from theta by an integral multiple of pi. So, this is a consequence of e to the i times an integral multiple of 2 pi being the same as you get back one.

So, therefore, you have this freedom to add any you know integer times 2 pi to theta. So, while e to the u of x comma y must be equal to 0, v of x comma y is equal to theta plus 2 n pi, where n can be any integer.

So, therefore, you know this function f of z we can think of as log of r. So, log of r, where r is a positive real number is of course, a familiar concept right. So, it is the natural logarithm of a real positive number. And, then so, there is an imaginary part; so, this imaginary part is you know there is some ambiguity around this right. So, it is theta plus 2 n pi where n can actually take infinitely many values.

So, in fact, you know this inverse function that, you know we should define, seems to have actually infinitely many values for a given z, you can find infinitely many complex numbers such that e to the f of z equal to z. So, you do not get just one answer, but. In fact, you will get infinitely many of these.

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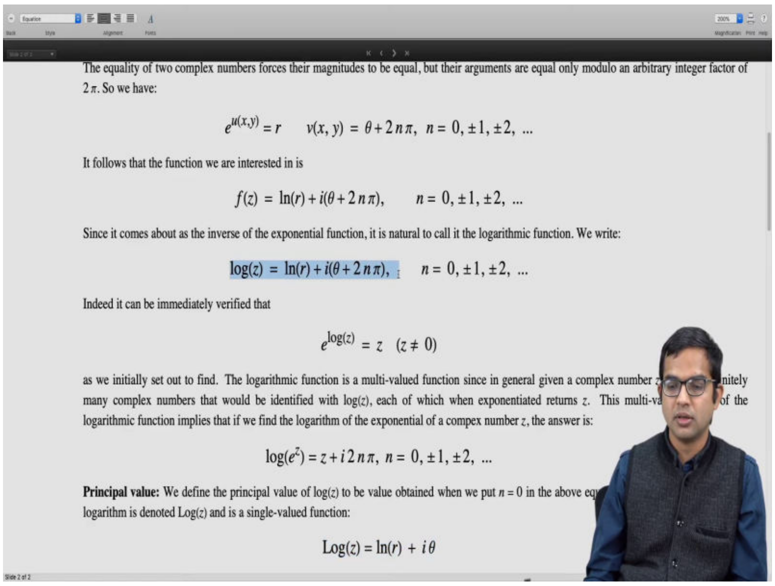

So, this is something which we can define as the logarithmic function. So, we write log of z, as log z is called a multivalued function is log r plus i times theta times 2 n pi, where n can take any of these values 0 plus or minus 1 plus or minus 2 and so on.

So, it may be verified that, you know e to the log z is equal to z as we decide of course, z cannot be allowed to be 0, because log of 0 is not defined right. So, you can take r to be very small and positive, but exactly at r equal to 0 this function has you know a singularity it is not defined at this point.

So, we seem to have managed to find the inverse of the exponential function, namely log z it can be defined in this manner right. It has this multi valued nature, which makes it somewhat tricky.

So, log of e to z. So, one aspect of this is although it is an inverse function. So, e to the log z indeed is equal to z, but if you do you know go in the other direction. If you try to find the log of e to the z, the log of a complex number is not unique right.

The way we have defined it so in fact, you could get z, but you also have many other answers around it right. So, which are all basically you know z, but with plus some integral multiple of you know i times 2 pi right.

So, you have an extra complex part, you have an extra imaginary part which comes in for free right. So, this is a little different from the kind of logarithm of a real number that we are familiar with. So, the moment you take the log of a complex number, there are these difficulties all of which basically stem from the multi valued nature of the log function.

So, it is convenient to define something called the principal value. So, this is analogous to how we define the principal value of the argument function. So, in fact, you know this multi valued nature of log function can be traced to the multi valued nature of the argument function right. So, this is how we got this ambiguity in v comes from the fact that the argument of a complex number has this there is this extra freedom to n pi freedom, that is where the extra freedom for the log function also comes in.

So, it is convenient to define the so-called principal value for the log function and so, when you refer to the principal value, we denote this as capital Log of z right. So, this is a single valued function and so, this refers to the so-called principal branch right. So, when you put n equal to 0 then so, this definition with n equal to 0 is called the principal value of the logarithmic x function.

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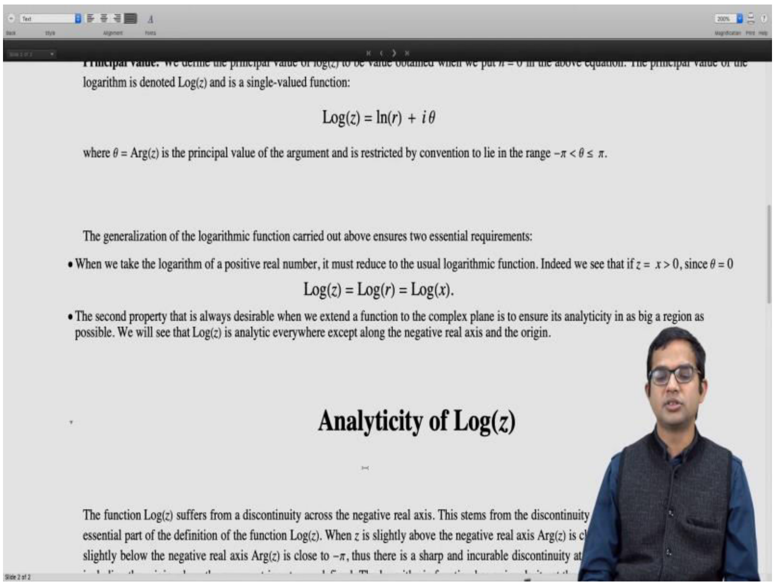

So, log of capital Log of z is equal to log of r plus i theta right, where now theta is this capital R of z right.

So, this is also similar to how we can think of a small a r g and a capital A r g and where capital A r g is the principal value of the argument. And, the convention we are following here is to restrict this argument to lie between minus pi and plus pi including plus pi right. So, in general of course, it can take you know, it can range from any you know alpha to alpha plus 2 pi right.

So, but I mean it is convenient for us to choose this so-called branch to lie between minus pi and plus pi, as when we are thinking about the principal value. So, the generalization of this log function is also carried out keeping in mind these two requirements. So, I mean of course, we looked at it as an inverse of the exponential function, but indeed. So, analyticity of this function is also you know assured in a very large region.

So, we will discuss analyticity of this log function in a moment, but first of all let us point out that this log function will indeed reduce the usual logarithmic function, if you have z equal x greater than 0 right. So, you have a real positive number, it must get back to us the same logarithm that we already are familiar with and you know if you are dealing with capital L o g, it is just you know log r as we know it right.

So, I mean even for real positive numbers. So, there is extra freedom associated with the log value. So, there is the complex part which you know we usually ignore right.

So, in fact, we are used to using just the log of r as a numerical value, there is no complex part, there is no imaginary part, that we associate with the log of a positive real number right, but if you look at this small log. So, the generalized notion of the log will actually be also multi valued you know for any complex number z right.

So, this is something worth emphasizing. So, the other property which is desirable is to ensure that this log function is analytic in as large a region as possible. So, it turns out that when we are looking at this capital Log of z, namely the principal value, we have restricted this capital Arg to lie between minus pi and plus pi.

So, there is one whole line of points which starts from the origin and goes all the way up to minus infinity along the negative real axis, which is somewhat problematic in the sense that this function log of z is not even continuous at that point right. So, this is something that cannot be avoided.

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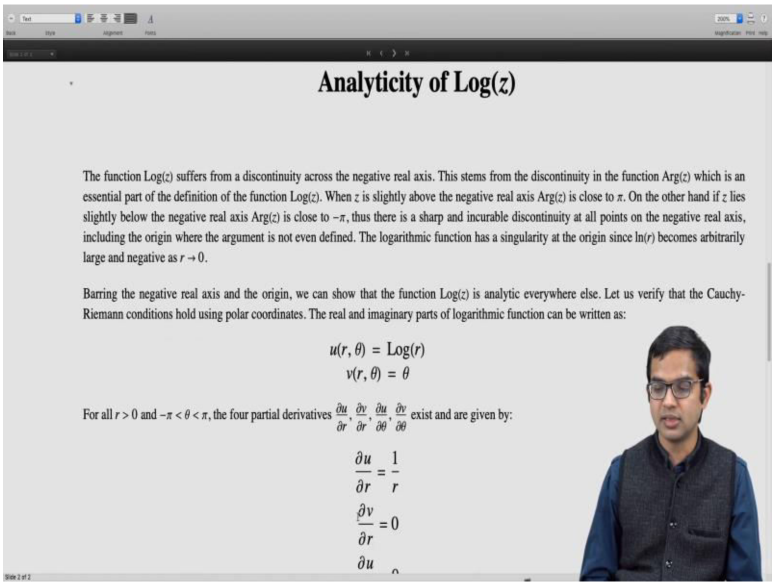

So, let us look at the analyticity of this log of z. And, so, if you are on the negative real axis there is a discontinuity. The reason is that this argument of z has a discontinuity as you cross the negative real axis. So, this is our choice of you know what is called the branch that we are looking at right.

So, your argument goes from minus pi to plus pi. So, if you are slightly below the negative real axis, then the argument is going to be close to minus pi, but if you are slightly above the negative real axis, the argument is going to be plus pi.

So, as you cross this line you are going to have a discontinuity right. And, therefore, the log function capital Log also has a discontinuity on that line and also at the origin right. In the limit of small r becoming very small and going to 0, it becomes very large negative and you have a logarithmic singularity at that point.

So, all points from the origin all the way up to negative infinity are you know there is discontinuity for the log function it has a discontinuity. So, there is no question of it having a derivative at that point right. So, we have seen that differentiability is a stronger condition than continuity.

So, if it is not even continuous, there is no question of differentiability and if it is not differentiable there is no question of analyticity. But, this log function is seen to be analytic everywhere other than this one ray, which starts from the origin goes all the way up to minus infinity, but everywhere else the log function is indeed analytic. As we can verify you know the Cauchy Riemann conditions will hold.

So, if you look at the real and imaginary parts of the log function, ok. The principal value you can write it as u of r comma theta is just Log of r, and v of r comma theta is just theta, so, for all r greater than 0 and minus pi less than theta less than pi right.

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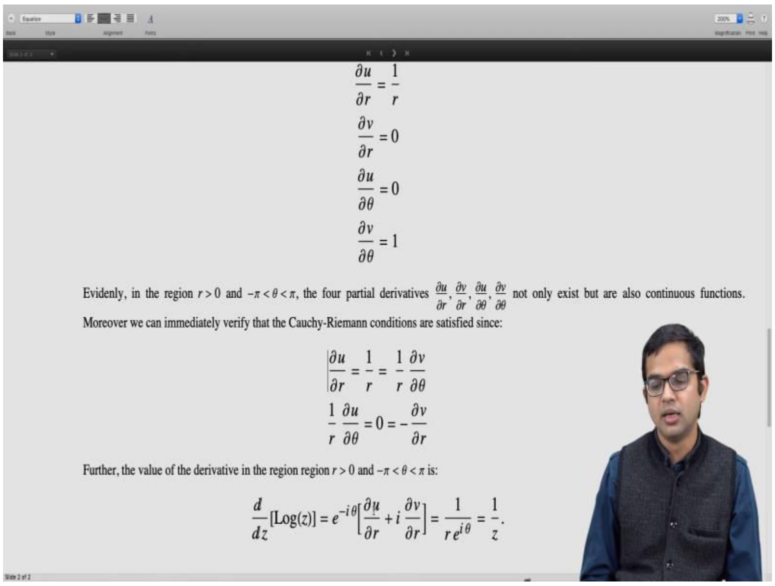

So, now theta equal to pi is not included everywhere other than this you know line, which goes from 0 to minus infinity. These four partial derivatives dou u by dou r, dou v by dou r, dou u by dou theta and dou v by dou theta exist and they are given by these nice smooth functions.

This is one over r 0 0 and 1 so, in fact, they are extremely well behaved, so, not only do these four partial derivatives exist in this region, but they are also continuous functions. So, everything that we desire you know for these kinds of the real part and the imaginary part hold and also the Cauchy Riemann conditions are also satisfied, which we can immediately verify in polar coordinates dou u by dou r is the same as 1 over dou v by dou theta both of them are equal to 1 over r. You know which comes from here; from these equations also 1 over r times dou u by dou theta is the same as 0, which is also the same as minus dou v by dou r.

So, both the Cauchy Riemann conditions hold and these nice properties that we demand for these functions u of r comma theta and v of r comma theta also hold, in this entire region r greater than 0 and theta mod of theta less than pi.

Therefore, the analyticity of the log function in this entire region is guaranteed and so, the derivative exists; the value of the derivative is something that we can immediately compute. We use the value of the derivative you know in polar coordinates, it is just e to the minus i theta and dou u by dou r plus i times dou v by dou r.

But, dou u by dou r and dou v by dou r are you know dou v by dou r is just 0 dou u by dou r is 1 over r. So, you have 1 over r times e to the i theta, which is the same as 1 over z right. So, the derivative of the log function is 1 over z you know which also nicely generalizes from the corresponding derivative for the log function of a real variable right.

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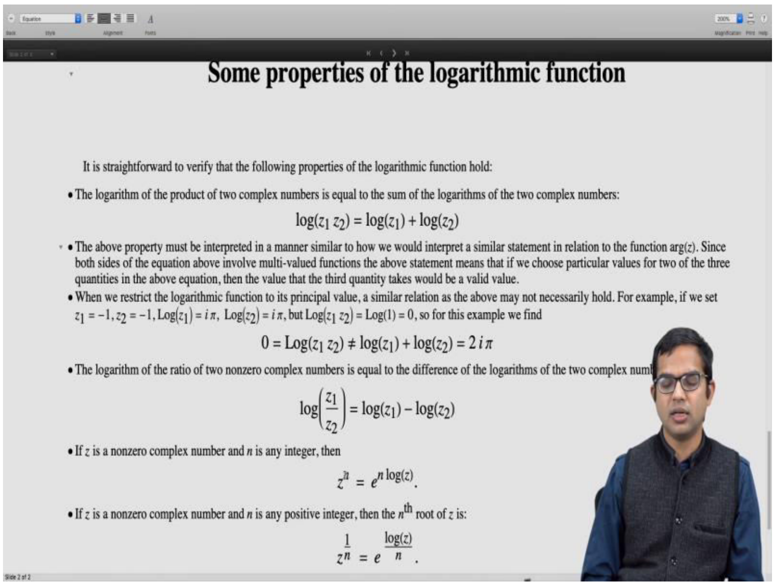

So, some of the properties of the log function let us quickly state them. So, if you take any two complex numbers, multiply them and take the log of this product, then the log of this product is the same as the sum of the logarithms of each of these complex numbers.

So, we must point out that you know each of these functions here, all each of these quantities here in the left hand side there is one quantity and on the right hand side there are two quantities. So, they are all multi valued. So, the way to interpret this statement is the way we interpreted a similar statement, when we had arguments right.

So, the argument of z 1 times z 2 is equal to the argument of z 1 plus argument of z 2 where you are using a small a right. And, so, the way to interpret this is to say that if you take one value for this quantity and one value for this quantity. And, if you add them together you are going to get a value, which is going to be one of the values of this quantity right.

So, it is going to be a valid answer no matter which value you take for here, which value you take for here for sure the result is going to be valid, the value that this function is going to take. So, that is the way to interpret this, but when we take the principal value this condition may not always hold.

So, one example is suppose you take z 1 equal to minus 1 and z 2 equal to minus 1. So, log of capital Log of z 1 the principal value is i pi and principal value of Log of z 2 is also i pi, but Log of z 1 times z 2 is so, it is just plus 1. So, which is Log of 1, which is 0. So, for this example we find that you know one of them will give you 0, but the other one will give you 2 pi i.

So, indeed this property does not necessarily hold if you have, if you have complex, if you are working with the principal value. But, I mean the more general logarithmic function indeed has this property, the logarithm of the ratio of two complex numbers you know also has a similar result right.

So, what you can think of 1 over z 2 is another complex number and then basically, it is really the same rule log of z 1 over, z 2 is equal to log 0 minus log z 2.If z is some non zero complex number and n is an integer then z to the n is going to be e to the n times log z right. So, this is also something which you know straightforwardly follows from the definition of the log function.

And, so, if in, if n were not an integer, but 1 over n integer then. So, you are looking at the nth roots of a complex number z to the 1 over n. So, although log z has you know there is some ambiguity about what its value is, because you have all this you know your many complex numbers, which this corresponds to, but when you take the exponential of n times log z. So, it is going to be n times 2 pi i here and therefore, when you take the exponential all of them are going to give you the same answer it does not matter which value of log z you take.

But, as far as this equation is concerned there is a unique value both on the right hand side and on the left hand side. But, when you take the nth root of a complex number so, z to the 1 over n, now you have an exponential of log z over n. And, so, now, because you have this denominator n sitting here if you have i times 2 pi k. So, there are going to be n distinct values for this right.

So, this is something that you can verify. After the nth value then they start repeating right. Because there is this denominator and then, if you add some integer multiple of 2 pi i times 2 pi. So, the exponential of such a quantity will just give you back one after point. So, there are exactly n distinct values for this function and that is also reflected in this quantity on the right hand side right.

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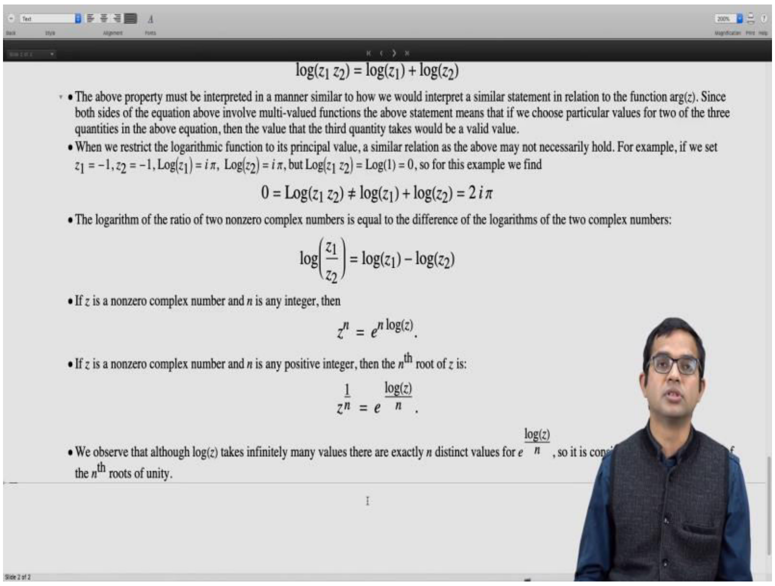

So, although log z itself can give you infinitely many different values whereas the nth roots of a complex number are exactly n, as we have already seen we worked out the nth roots of unity. And they all sit on the circle and they are all equidistant from each other on the circle and so, they correspond to different phases right. So, I mean it is exactly like that and when you have some other complex number you take its magnitude and pull out the magnitude, and then you are really working with the nth roots of unity ok.

So, in this lecture we have introduced the log function and we have also talked about how the log function is in general a multivalued function, and how we can consider the principal value of it, and we have also seen how the log function is analytic everywhere except, you know in a line which starts from the origin and goes all the way up to minus infinity ok. That is all for this lecture.

Thank you.