Mathematical Methods 2
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Module - 02
Complex Variables
Lecture - 20
Inverse Trigonometric and Hyperbolic Functions

Ok. So, we have seen how Trigonometric functions and Hyperbolic Functions generalize when we have Complex Variables. So, in this lecture, we will look at how to find inverses of these functions.

So, the problem we consider is suppose we are interested in finding the value of z which you know whose sine is a certain complex number whose sine or a cosine or whichever trigonometric function you are interested in or a hyperbolic function is it possible to work backwards.

And so, it turns out, so it is possible to find such a complex number. But in fact, in general there are going to be lots of complex numbers all of whose sines are the same, right. So, we will end up with what are called multivalued functions. And these are all you know connected to the log function and its multivalued nature, right. So, it is some of these properties that we discuss in this lecture, ok.

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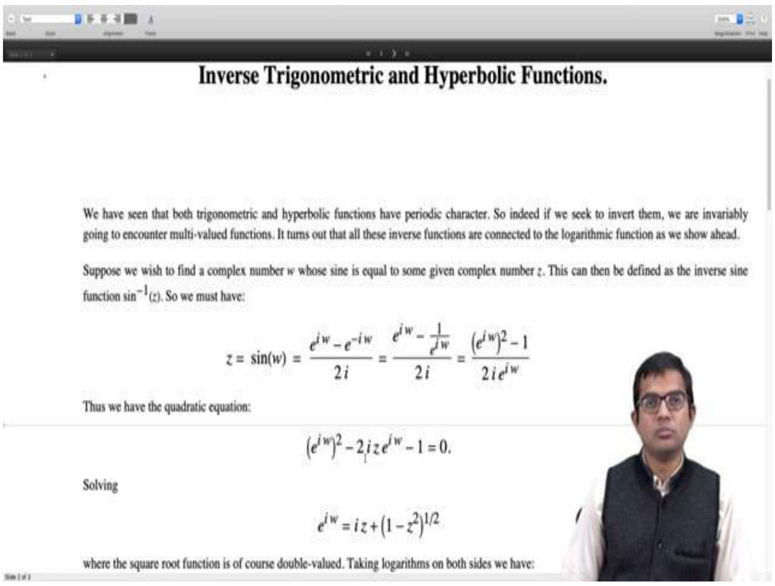

So, suppose we want to invert you know the sine function. So, in other words, we wish to find a complex number w to sine is equal to some complex number z, right. So, we want to find the sine inverse of z. And so, we can call this as the sine inverse of z, right. So, we are interested in finding a w, such that sin of w is equal to z, right.

So, but the sine of w is nothing but e to the i w minus e to the minus i w divided by 2 i. And we can rewrite into the minus i w as 1 over e to the i w, and then we can you know collect these terms in an appropriate way, so which is you get e to the i w the whole squared minus 1, then divided by 2 i e to the i w. So, thus we have actually a quadratic equation.

So, if you bring this e to the 2 i e to the i w to the left hand side you have a z here, and so, you bring all terms together onto one side. So, then you have e to the i w the whole squared minus 2 i z times e to the i w minus 1 equal to 0. So, this is a quadratic equation in e to the i w, right. So, w is an unknown, so e to the i w is an unknown, but this is really just a quadratic equation in e to the i w which we know how to solve, right.

So, the solution is nothing, but you know minus b, in this cases 2, 2 i z. So, plus or minus, so we do not have to explicitly write down plus or minus because we understand that this function is a multivalued function, right. So, to the power half means both plus root and negative root, both roots are allowed.

So, we have you know 1 minus z squared the whole power half you know there is an overall 2 which comes out and then you divide by 2, right. So, you can work this out carefully. So, and convince yourself that indeed. This is e to the i of w is just i z plus 1 minus z squared the whole power half, right. I emphasize again is to be thought of as is really a multivalued function, there is a plus sign or a minus sign both values are allowed.

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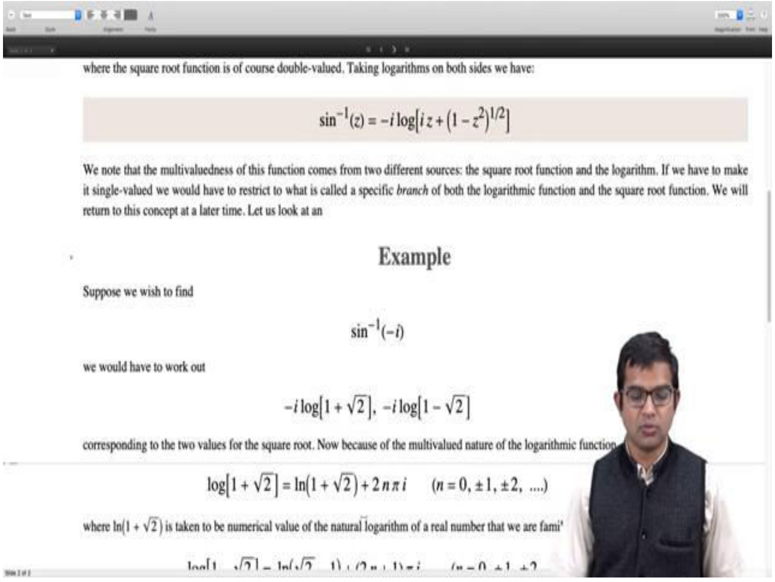

So, already there is a multivalued nature to this. And now we are also going to have to take a log on both sides. So, in fact, to find w, right, so we have to take the log of this whole stuff and then divides throughout by i and that is give going to give us a minus i times log of i z plus 1 minus z squared of a power half.

So, sine inverse of z is defined to be this function, and so, immediately we see that there are actually two sources of multivaluedness, this kind of a definition of an inverse function. So, inverse functions are naturally endowed with this kind of multivalued nature, right.

So, I mean this is you know relates to the fact that the function that you started with itself has a periodic character, right. So, there is a repetition of a certain value which keeps coming back and therefore, indeed when you go to when you invert the function of this guy surely there is going to be multivaluedness.

And so, the square root is you know is one kind of it is a; it is a by valued nature for this function, and then logarithm function actually gives you infinitely many values, right. For a given z there are infinitely many values of which correspond to sine inverse of z, and this infinite multivaluedness comes from the log function, right.

So, there is a way to make the single valued by considering what is called a specific branch of each of these functions, right. For example, we can choose the square root function we can take the positive square root. So, then we say it is now the positive branch. And for the log function we have seen that often it is useful to work with what is called principal value. Actually, if you look at it in that way then you know you can associate a particular value for the sine inverse of 0.

So, let us look at an example, where you know this multivaluedness of this function becomes apparent. Suppose, we wish to find the sine inverse of minus i, right, so then we would have to work out according to this formula, we would have to work out minus i log of 1 plus square root 2 and minus i log of 1 minus square root 2.

I have explicitly made you know the plus sign and minus sign separate, I mean we understand square root 2 to be plus square root 2, we are working with real numbers. So, these are the two you know values for sine inverse of minus i. And each of them has the logs, so that infinitely many different values for each of them, right. So, let us look at how is that a compact way to write down all these different values.

Now, so, the square root 2 of course, it comes from the fact that you know when I do 1 minus z squared, I have to do 1 minus z squared z is minus i. So, i squared is minus 1. So, that is going to give us root square root 2, right, so we have a 1 sitting here. So, indeed you know there are these two different values.

But let us look at log of 1 plus root 2 is actually nothing, but now it is convenient to call this lan of 1 plus root 2 to refer to, so the value that we ascribe to the logarithmic of a positive real number the usual value that we ascribe. But, in fact, there is also this freedom of adding 2 n pi i, right.

As we have seen it is a multivalued function. So, in general, if you if you write log of any number including a real positive number, it is not just the usual number that we ascribe to logarithm of a positive number, but in fact, you can add 2 n pi i for free, when n can be any integer, right. So, this is what we mean by this quantity. So, let us also try and work out exactly what we mean by log of 1 minus root 2.

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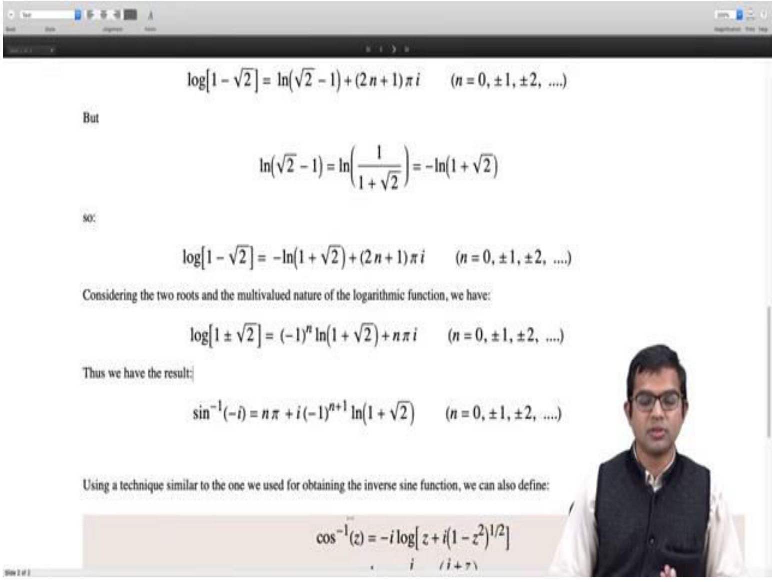

So, if we yeah, look at this quantity log of 1 minus root 2, then, in fact, lan of positive real number which is the modulus of this this quantity should be taken, which is root 2 minus 1 root is greater than 1. And then, I mean there, so you can think of this as log of root 2 minus 1 times minus 1, and then you apply you know the rule that log of some quantity times some other quantities log of the first quantity times of plus the log of the second quantity which log of minus 1 will be just 2 n plus 1 times pi i.

In fact, even here you can think of this as log of 1 plus root 2 times 1 which is ln of 1 plus root 2 plus log of 1, and then log of 1 is actually 2 n pi i, it is not just 0, right. So, you have, it is a multivalued function. So, but so, we can also if you look at this quantity ln of root 2 minus 1, it is something which we can rewrite in the following way ln of root 2 minus 1 is actually nothing but ln of 1 plus root 2, right.

So, you can verify this. And then 1 over 1 plus root 2, log of this, ln of this is actually nothing, but minus ln of 1 plus root 2. So, in fact, there is a way to rewrite this back in a form similar to the first quantity. So, log of 1 minus root 2 is, in fact, nothing but minus log of 1 plus root 2 plus 2 n plus 1 pi i, right, where n can take all integer values.

So, you get all these odd factors of pi i here and you can even factors of pi i here. And so, you have this same quantity here, and same quantity here, except that now you have a minus sign. So, we can combine both of these together and write it in this form, right. And allow you know instead of saying 2 n plus 1 times pi i 2 n times pi i you can just say n pi i and allow all integers and just take care of the sign change odd n. If n is even then it must be plus and if n is odd it must be minus.

So, in fact, there is this compact we have writing log of 1 plus or minus square root 2. And then, we can go ahead and plug this back into the original, you know multivalued, two different values that we gave and each of them themselves come with all these free degree of freedom in terms of n.

So, in fact, you have this final result sine inverse of minus i, right, we should recall that there is this minus i sitting also which we need to tag along. So, sine inverse of minus i is actually n pi, so minus i times n pi i, this is going to be sine pi. Then minus 1 to the power n times minus 1, it is going to become minus 1 to the n plus 1, and then we also have this i times log of 1 plus root 2, it is a natural logarithm of this real positive number 1 plus root 2, right.

So, we see that it is quite subtle, in fact, to take the inverse function, sine inverse is one example we have looked at. But, in fact, we can use the similar method to define inverse, you know other inverse functions. So, cosine inverse of z. You can work this out, using a method very similar to the sine inverse function, you know other quadratic equation, solve for it.

And you can show that it makes sense to define cosine cos inverse of z to be minus log of z plus i times 1 minus z squared the whole power half, and tan inverse of z is defined as i over 2 log of i plus z divided by i minus z, right.

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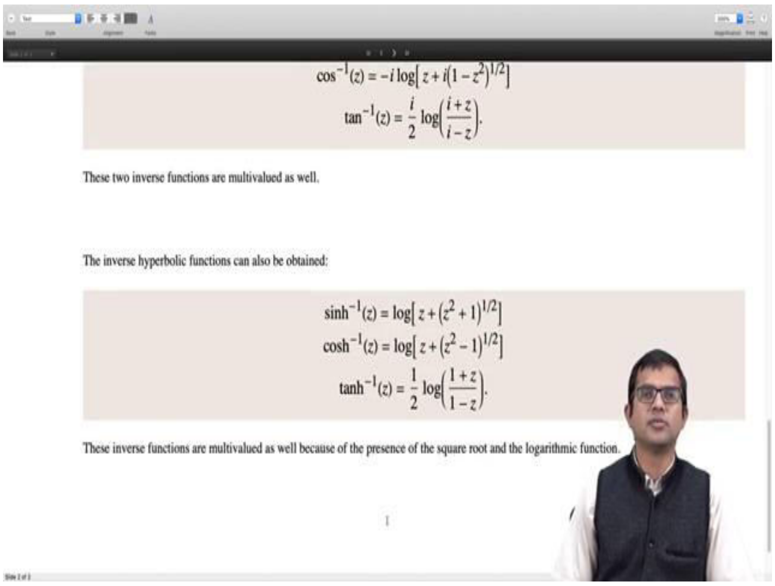

Both of these are again multivalued functions. And there is a way to analyze you know these functions as well, inside some single valued branch as it is called. And, in fact, within that branch since log is a nice well behaved analytic function. In fact, all these inverse functions are also going to be nice and well behaved within so called single valued branch, right.

So, which so, we may discuss this aspect a little more carefully at a later time. But for our purposes at this point we just want to be aware like it is possible to define in a meaningful way all these inverse functions, and but they have this multivalued character.

So, ultimately all of this multivalued character goes back to these log functions or you know square root and so, on. So, these are you know the, and also the arg function which in turn is also hidden deep under underneath all these multivaluedness of these functions, ok.

So, we quickly point out that the inverse hyperbolic functions can also be obtained you know by similar methods. So, after all you know hyperbolic functions and trigonometry functions that intimately connected as we have already seen. So, one can define the inverse hyperbolic sine function, sine in h inverse of z to be log of z plus z squared plus 1 over the power half, and then cosh inverse of z is log of z plus and z squared minus 1 to the power half, and tanh inverse of z is half log of 1 plus z divided by 1 minus z.

So, and indeed these functions to have their multivalued character, ok. So, that is all for this lecture.

Thank you.