

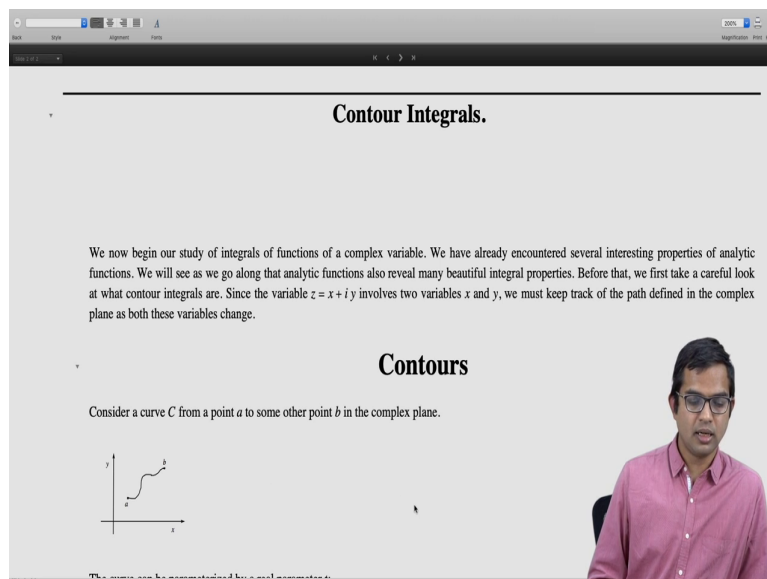
**Mathematical Methods 2**  
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**Complex Variables**  
**Lecture - 22**  
**Contour Integrals**

So, we have spent a fair amount of time looking at analytic functions and their properties in relation to differentiation. So, with this lecture we begin our discussion of integral properties of functions of a complex variable.

So, we will first of all look at how to define the notion of a contour integral in this lecture. So, we will first describe what a simple curve is like and how to think of integration along a path in the complex plane - that is the subject matter for this lecture.

(Refer Slide Time: 01:05)



The image shows a presentation slide with a white background and black text. At the top, the title "Contour Integrals." is centered. Below it, a paragraph of text reads: "We now begin our study of integrals of functions of a complex variable. We have already encountered several interesting properties of analytic functions. We will see as we go along that analytic functions also reveal many beautiful integral properties. Before that, we first take a careful look at what contour integrals are. Since the variable  $z = x + iy$  involves two variables  $x$  and  $y$ , we must keep track of the path defined in the complex plane as both these variables change." Below this text, the word "Contours" is centered. Underneath, another paragraph says: "Consider a curve  $C$  from a point  $a$  to some other point  $b$  in the complex plane." To the left of this text is a small diagram of a Cartesian coordinate system with a wavy curve starting at point  $a$  and ending at point  $b$ . On the right side of the slide, there is a video feed of a man with glasses and a pink shirt, who is the lecturer.

So, there are two different independent variables  $x$  and  $y$  when you are looking at a complex variable  $x$  and  $y$ . So, we have to come up with a suitable notion of a path, if you want to do an integration and both  $x$  and  $y$  are changing. So, how can we think of an integration of  $f$  of  $z$   $dz$ ? We have to come up with a suitable way to define what such an integral would be. And so the starting point for this is to define the notion of a contour.

So, you consider some curve from a point a to some other point b in the complex plane right. I have sketched some curve I have two different points a and b and then I can make up some arbitrary curve.

(Refer Slide Time: 01:51)

The curve can be parameterized by a real parameter  $t$ :

$$x = x(t), \quad y = y(t) \quad ; \quad t_a \leq t \leq t_b$$

where  $t_a$  and  $t_b$  are the values of the real parameter at the end points. We assume that  $t$  increases monotonically as the curve is traversed from  $a$  to  $b$ , so that  $x(t)$  and  $y(t)$  are real single-valued functions of  $t$ . If  $x(t)$  and  $y(t)$  are continuous functions of  $t$ , the curve  $C$  is called an **arc**. It is convenient to think of this curve as a continuous sequence of complex numbers:

$$z(t) = x(t) + i y(t) \quad t_a \leq t \leq t_b$$

The arc is called a **simple arc** if it does not cross itself. Thus every point  $z = (x, y)$  has a unique  $t$  associated with it. When the arc  $C$  is simple except for the fact that  $z(t_b) = z(t_a)$ , we say that  $C$  is a **simple closed curve**.

**Example**

The curve

$$z = z_0 + R e^{i\theta} \quad ; \quad 0 \leq \theta \leq 2\pi$$

is a simple closed curve oriented in the counterclockwise direction. It is a circle of radius  $R$  centred about the point

$$z = z_0 + R e^{-i\theta} \quad ; \quad 0 \leq \theta \leq 2\pi$$

Now, this curve could be parameterized by some real parameter  $t$ . In fact, think of this parameter  $t$  as some time - the curve is generated as a function of time. You start at time  $t$  equal to 0 or  $t$  equal to  $t_a$  in this case and go up to time  $t_b$  and maybe you know measure the time as you draw a curve of this kind.

So, typically we are interested in curves which have some nice properties which we will just state up front. So,  $t_a$  and  $t_b$  are real numbers. They are the values of the real parameter at two end points, we assume that  $t$  will increase monotonically. It's helpful to take  $t$  as time and it keeps increasing and both  $x$  of  $t$  and  $y$  of  $t$  are changing as a function of time.

So, you can think of this complex number which is made up of  $x$  and  $y$  both of these real parts and imaginary parts themselves are changing as a function of time and we will look at a scenario where both  $x$  of  $t$  and  $y$  of  $t$  are continuous functions of time right.

And so, then the curve is called an arc, it is convenient to think of you know these curves are continuous sequence of these complex numbers like I just said and this arc would be called as simple arc if it does not cross itself right.

So, we do not consider curves where there are lots of criss-crosses and so on are not convenient for the purpose of contour integrals which we will look at in some detail as we go along.

So, simple arcs are useful for us and then we make this one exception where you know if your curve comes back to where it started then it is called a simple closed curve right. So, there is just exactly one point where you know the  $z$  of  $t_b$  is equal to  $z$  of  $t_a$  right. There are two different  $t$ 's for which you get the same  $z$  that is right at the initial point and right at the end point and such a curve would be called a simple closed curve.

So, in general every point through which your curve passes has a unique  $t$  associated with it, you cannot find two different  $t$ s at which this curve may reach the same point.

So, let's look at an example. So, if you consider a path like this  $z$  is equal to  $z_0$  plus  $R$  times  $e$  to the  $i$  theta and then you. In fact, theta is like a  $t$  here right. So, theta is changing from 0 and going all the way up to  $2\pi$  it is a simple closed curve right you. And its basically it traces out a circle whose centre is  $z_0$  and whose radius is  $R$  and you start at theta equal to 0 and then you know you complete one circle.

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The slide is titled "Example" and contains the following text and equations:

The curve

$$z = z_0 + R e^{i\theta} \quad ; \quad 0 \leq \theta \leq 2\pi$$

is a simple closed curve oriented in the counterclockwise direction. It is a circle of radius  $R$  centred about the point  $z_0$ . If we consider the curve

$$z = z_0 + R e^{-i\theta} \quad ; \quad 0 \leq \theta \leq 2\pi$$

it is also seen to be a simple closed curve again representing a circle of radius  $R$  centred about the point  $z_0$  but it is oriented in the clockwise direction. On the other hand the curve

$$z = z_0 + R e^{i2\theta} \quad ; \quad 0 \leq \theta \leq 2\pi$$

traverses exactly the same set of points as in the above two curves but it is different from either of them because it traverses the circle in the counterclockwise direction.

An arc is called a **regular arc** if the parametric derivatives  $x'(t)$  and  $y'(t)$  exist and are continuous with  $x'^2(t) + y'^2(t) > 0$ . Thus a regular arc is a smooth curve and a unique point on the complex plane corresponds to every value that  $t$  takes.

We also work with contours that are **piecewise regular**, which consist of a finite number of regular arcs. We

Slide 2 of 2

But if you consider and it is the orientation is in the clockwise direction right because 0 its going from 0 to 2 pi, but if you looked at z equal to z naught plus R times e to the minus i theta and again you let theta go from 0 to 2 pi. Now again you will get a circle, but it is in the clockwise direction right.

So, on the other hand if you consider something like z equal to z naught plus R times e to the i 2 theta and you let it run from 0 to 2 pi theta. So, then this is also a circle of radius R centred about z naught and it is in the counterclockwise sense, but this curve manages to go around 2 times, it looks around twice right. So, I mean apparently the same type of curve means slightly different things depending upon you know the details right. So, this is important to keep it in mind.

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traverses exactly the same set of points as in the above two curves but it is different from either of them because it traverses the circle twice in the counterclockwise direction.

An arc is called a **regular arc** if the parametric derivatives  $x'(t)$  and  $y'(t)$  exist and are continuous with  $x'^2(t) + y'^2(t) \neq 0$  at any point on the arc. Thus a regular arc is a smooth curve and a unique point on the complex plane corresponds to every value that  $t$  takes.

We also work with contours that are **piecewise regular**, which consist of a finite number of regular arcs. We also work with a **simple closed contour** when only the initial and final values of  $z(t)$  are the same:  $z(t_a) = z(t_b)$ .

**Example**

The curve

$$z = \begin{cases} t + it & 0 \leq t \leq 1 \\ (2-t) + i & 1 \leq t \leq 2 \\ (3-t)i & 2 \leq t \leq 3 \end{cases}$$

is a piecewise regular simple closed curve oriented in the counterclockwise direction.

Now, an arc is called a regular arc if  $x$  of  $t$  and  $y$  of  $t$  are continuous and both  $x$  prime of  $t$  and  $y$  prime of  $t$  exist and they are continuous and also  $x$  squared plus  $y$  prime squared of  $t$  is not equal to 0 at any point on the arc right.

It seems like a weird requirement. But actually if you pause for a moment what this simply means is you know the only way for the square of the sum of 2 squares to be 0 is each of them is separately 0. And such a scenario is something which we do not want to allow right.

What would that mean if both  $x$  prime of  $t$  and  $y$  prime of  $t$  are both 0? That means, that you know there is a point at which neither  $x$  nor  $y$  is changing. So, in some sense your curve is moving along and then at some point it just keeps on staying there for some time that is the type of scenario which you do not want to allow right.

So, there is a continuity associated with it and you know  $x$  prime and  $y$  prime of  $t$  both cannot be simultaneously 0 at any point on this arc and then it is called a regular arc. There is this smoothness associated with regular arcs which is valuable when we come up with this definition of a contour integral.

So, we also work with contours which are piecewise regular right. So, there will be these sort of points where you know your curve suddenly undergoes a change and, but within every segment of the path that your contour is taking you know this regularity is maintained, but

there are these in between points were you know, these conditions do not quite hold and still we can work with such curves as well. So, it is also useful to work with piecewise regular contours.

So, simple closed contour is when you know  $z$  of  $t_a$  is equal to  $z$  of  $t_b$ . So, the starting point and the end point are the same, closed contours are going to be very important right. So, we definitely need that notion.

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We also work with contours that are **piecewise regular**, which consist of a finite number of regular arcs. We also work with a **simple closed contour** when only the initial and final values of  $z(t)$  are the same:  $z(t_a) = z(t_b)$ .

**Example**

The curve

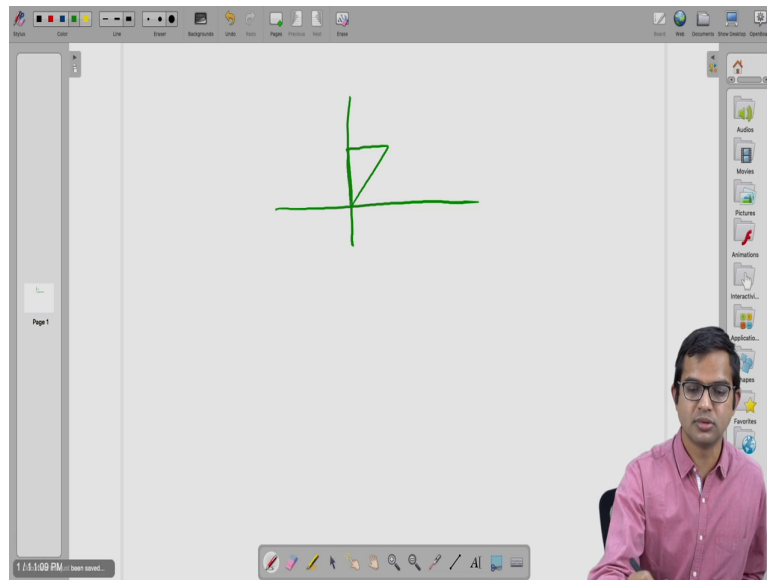
$$z = \begin{cases} t + it & 0 \leq t \leq 1 \\ (2-t) + i & 1 \leq t \leq 2 \\ (3-t)i & 2 \leq t \leq 3 \end{cases}$$

is a piecewise regular simple closed curve oriented in the counterclockwise direction.

*(Note: The diagram in the slide shows a triangle in the complex plane with vertices at (0,0), (1,1), and (2,1). The path goes from (0,0) to (1,1), then to (2,1), and finally back to (0,0).)*

So, let us look at an example. So, if I consider the curve  $z$  is just  $t$  plus  $i t$ , when you go from  $t$  going from 0 to 1 and then  $2 - t$  plus  $i$  you know  $1 \leq t \leq 2$  and  $3 - t$  times  $i$ ,  $2 \leq t \leq 3$ . So, this is a piecewise regular simple closed curve right, it is oriented in the clockwise direction.

(Refer Slide Time: 08:19)



So, you can convince yourself that this is actually nothing but you know you have  $a$ , you know this is the complex plane. So, the first part of this journey looks like this and then it comes back around and then it comes back around right. So, that is what this term is which I have also traced here.

So, you start from  $0$  and it goes up to this point along this  $45$  degree angle you reach  $1 + i$  then you come back along this direction which is parallel to the  $x$  axis and hit the  $y$  axis and then you come back down and it is  $a$ . So, it is piecewise regular we can also work with such curves.

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**Contour integrals.**

Consider a function  $f(z)$  of a complex variable defined over a region which contains a contour  $C$  that is piecewise regular given by  $z(t)$ . We consider the function along the contour  $C$

$$f(z) = f(z(t)).$$

Suppose we consider two nearby points on the contour  $C$ , represented as  $z$  and  $z + \Delta z$  where

$$\Delta z = \Delta x(t) + i \Delta y(t).$$

In the limit  $\Delta z \rightarrow 0$ ,  $\Delta t \rightarrow 0$ , since  $\sqrt{x'^2(t) + y'^2(t)} \neq 0$ , we can write:

$$dz = \frac{dx(t)}{dt} dt + i \frac{dy(t)}{dt} dt.$$

On the other hand, we can write the function  $f(z)$  in terms of its real and imaginary parts as:

$$f(z(t)) = u[x(t), y(t)] + i v[x(t), y(t)] = u(t) + i v(t).$$

Therefore

So, let's define the notion of a contour integral, suppose we consider this function  $f$  of  $z$  and a path. So, there is a contour  $C$  that is piecewise regular and it is given by  $z$  of  $t$ . So, we are looking at a function  $f$  of  $z$  of  $t$ . And then we want to see what happens, how does  $z$  change when you change  $t$  by a small amount the parameter. The real parameter changes by a small amount and so there is going to be a change in  $\Delta x$  which we call  $\Delta x$  of  $t$  and change in  $y$  which is  $\Delta y$  of  $t$ .

So, as you take the limit  $\Delta t$  becoming very small and therefore,  $\Delta z$  also becomes very small, but since we have also used this condition you know  $x'$  squared of  $t$  plus  $y'$  squared of  $t$  is not 0 and it is a positive quantity. So, square root of this is non zero. So, what it means is this modulus of this quantity is non zero. So, we can write you know  $dz$  is  $dx$  by  $dt$ ,  $dt$  plus  $i$  times  $dy$  by  $dt$ ,  $dt$ .

So, basically since this quantity is non zero. So, it is meaningful to think of this  $dz$  its not going to become you know, it is an infinitesimal quantity which is not itself 0 right. So, this is you know this is an important restriction to make which we already did when we introduced the notion of a regular arc or regular contour right. I mean of course, this is violated at these very special points right.



So, suppose you have a piecewise regular contour at these points its violated, but we for the moment let us not worry about you know how to make sense at these in between points. But in general if you have a piecewise, a regular contour you know this is well defined and it works out right.

So, in some sense it is a bit like how we play with piecewise continuous functions and within functions of a real variable. So, if you have this then  $f$  of  $z$  of  $t$  is  $u$  of  $x$  of  $t$  comma  $y$  of  $t$  plus  $i$  times  $v$  of  $x$  of  $t$  comma  $y$  of  $t$  we are writing out the real part and the imaginary part separately.

(Refer Slide Time: 11:25)

$dz = \frac{dx}{dt} dt + i \frac{dy}{dt} dt$

On the other hand, we can write the function  $f(z)$  in terms of its real and imaginary parts as:

$$f(z(t)) = u[x(t), y(t)] + i v[x(t), y(t)] = u(t) + i v(t).$$

Therefore

$$f(z) dz = \left[ u(t) \frac{dx(t)}{dt} - v(t) \frac{dy(t)}{dt} \right] dt + i \left[ v(t) \frac{dx(t)}{dt} + u(t) \frac{dy(t)}{dt} \right] dt$$

We define the contour integral of  $f(z)$  over the contour  $C$  as:

$$\int_C f(z) dz = \int_a^b \left[ u(t) \frac{dx(t)}{dt} - v(t) \frac{dy(t)}{dt} \right] dt + i \int_a^b \left[ v(t) \frac{dx(t)}{dt} + u(t) \frac{dy(t)}{dt} \right] dt$$

We observe that such a contour integral can thus be expressed in terms of line integrals over the contour  $C$  in the  $XY$  plane:

$$\int_C f(z) dz = \int_C (u dx - v dy) + i \int_C (v dx + u dy).$$

The contour integral of a function is generally dependent on the choice of the contour  $C$  and not just on the end points.

I

**Example**

And so then we can think of this  $f$  of  $z$  of  $t$  times  $dz$  as just the multiplication of two complex numbers  $f$  of  $z$  of  $t$  itself has a real part and an imaginary part,  $dz$  also has a real part and an imaginary part.

And so you can go ahead and multiply these two and so you will get  $u$  times  $dx$  by  $dt$  minus  $v$  times  $dy$  by  $dt$  you know the whole thing multiplied by  $dt$  plus  $i$  times  $v$  of  $t$   $dx$  by  $dt$  plus  $u$  of  $t$   $dy$  by  $dt$  also multiplied by  $dt$ .

So, we define this contour integral. So, now, we can put in this contour integral and then take this to be an integral in terms of  $dt$  right. So, it although this is just a function of  $t$  at every

point and this is just a real integral from  $a$  to  $b$  and. So, this information about the path that you are taking is actually embedded into this right.

So, because we have used you know this idea of what happens to  $z$  the small change in  $t$  gives you a corresponding small change in  $dz$  and that is already encoded into this. So, we see that in fact, this contour integral is a meaningful idea right if the path is you know has some reasonable restrictions that is the kind of paths we will be working with and then you have.

So, you have these two different integrals one for the real part and one for the imaginary part, which you can actually think of as two line integrals basically along these contours. So,  $u$  minus  $v$   $dy$  along this contour plus  $i$  times, along the same contour  $v$   $dx$  plus  $u$   $dy$ .

So, basically a contour integral is made up of these two line integrals, one for the real part and one for the imaginary part and which one can write down in terms of the real part of the function and the imaginary part of the function.

(Refer Slide Time: 13:29)

**Example**

Let us compute the contour integral

$$I = \int_C z^4 dz$$

along the right half-circle

$$z = 2e^{i\theta} ; \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

of radius 2 centred about the origin, starting from  $z = -2i$  and ending at the point  $z = 2i$ .

Since  $dz = 2ie^{i\theta} d\theta$  we have

$$\begin{aligned} I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2e^{i\theta})^4 \cdot 2ie^{i\theta} d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^5 e^{5i\theta} d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 32 e^{5i\theta} d\theta = 4\pi i. \end{aligned}$$

So, let us look quickly at a couple of examples of how this works out, suppose we want to compute the contour integral of  $z$  star  $dz$  along this contour which is this half, right half circle  $z$  is equal to 2 times  $e$  to the  $i$  theta right.

So, with theta going from minus pi by 2 to plus pi by 2. So, z starts from minus 2 i and goes along you know this positive direction and reaches plus 2 i. So, it is a circle of radius 2 and with origin being the centre. So, we see that if z is 2 times e to the i theta dz is going to be 2 times i times e to the i theta d theta.

So, we have I you know is this integral from minus pi by 2 to plus pi by 2, 2 times e to the i theta the whole star we have to do and then in place of dz we have 2 times i times e to the i theta d theta, which is simply given by you know just in place of 2 times e to the i theta. I have 2 times the whole star and 2 times e to the minus i theta and then I have 2 i e to the i theta. So, this e to the i theta e to the minus i theta they anneal into each other. And so, we have 4 times i d theta which is straight forward to complete. So, and we I just get 4 pi i.

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Since  $z z^* = |z|^2 = 4$  along the entire path, we can rewrite the above result as:

$$\int_C \frac{1}{z} dz = \pi i.$$

**Example**

Let us compute the contour integral of the same integrand as above but over a different path that starts from  $z = -2i$  and goes to  $z = 2i$ , along the straight line between the two points. We wish to compute:

$$I = \int_C z^* dz$$

along the straight line contour

$$z = it ; \quad -2 \leq t \leq 2.$$

Since  $dz = i dt$  we have

$$I = \int_{-2}^2 (it)^* i dt = \int_{-2}^2 -i t i dt$$

So, we also observe that along this entire path it is a circle. So, z times z star is just mod z squared which is 4 it is a circle of radius 2, so z times z star is 4. So in fact, I can think of z star as 4 over z. So, I can write this as contour integral of 4 over z dz over this contour is 4 pi i or equivalently I can say contour integral 1 over z over this path dz is the same as pi i right. So, this is completely equivalent to the result we just derived.

So, this is just you know half circle. So, we will quickly point out that if you had taken the same integrand, but over a slightly different path actually quite a different path. Suppose I

started from minus 2 i and if I go to plus 2 i along a straight line. So, then I get a different answer right.

So, this contour integral in general is a path dependent operation. So, if I consider the straight line contour I take z to be i t and it starts from minus 2 and it goes all the way up to 2 t. So, initially it is minus 2 i finally, it is plus 2 i and z is i t at all at in between points and dz is i times dt.

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along the straight line contour

$$z = it ; \quad -2 \leq t \leq 2.$$

Since  $dz = i dt$  we have

$$\begin{aligned} I &= \int_{-2}^2 (it)^* i dt \\ &= \int_{-2}^2 -it i dt \\ &= \int_{-2}^2 t dt = 0. \end{aligned}$$

Thus for this path with have:

$$I = \int_C z^* dz = 0.$$

$$I = \int_C z^* dz = 0.$$

So, integral is going to be minus 2 to plus 2 i t the whole star times i dt right in place of dz i write i dt and, but I t star is. So, t is just real. So, in place of i I have to put minus i then I have 1 i here. So, minus i squared is 1. So, I get t dt; but t dt is t squared over 2 plus 2 and minus 2 does not matter it is an even function.

So, it is in fact, 0. So, thus we see that if you take this path for the same integrand from the same initial point to the same final point by, but along a different path you get a different answer, right?

So, we will come back to you know such path dependent nature of this function and sometimes you know there is a path independence as well in certain context, but this is something that we will discuss later. That is all for this lecture.

Thank you.