Mathematical Methods 2
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Complex Variables
Lecture - 22
Contour Integrals

So, we have spent a fair amount of time looking at analytic functions and their properties in relation to differentiation. So, with this lecture we begin our discussion of integral properties of functions of a complex variable.

So, we will first of all look at how to define the notion of a contour integral in this lecture. So, we will first describe what a simple curve is like and how to think of integration along a path in the complex plane - that is the subject matter for this lecture.

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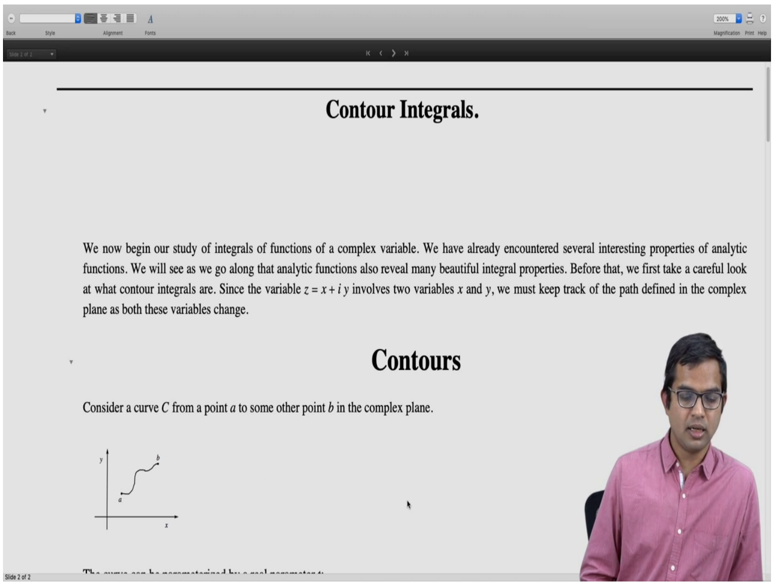

So, there are two different independent variables x and y when you are looking at a complex variable x and y. So, we have to come up with a suitable notion of a path, if you want to do an integration and both x and y are changing. So, how can we think of an integration of f of z dz? We have to come up with a suitable way to define what such an integral would be. And so the starting point for this is to define the notion of a contour.

So, you consider some curve from a point a to some other point b in the complex plane right. I have sketched some curve I have two different points a and b and then I can make up some arbitrary curve.

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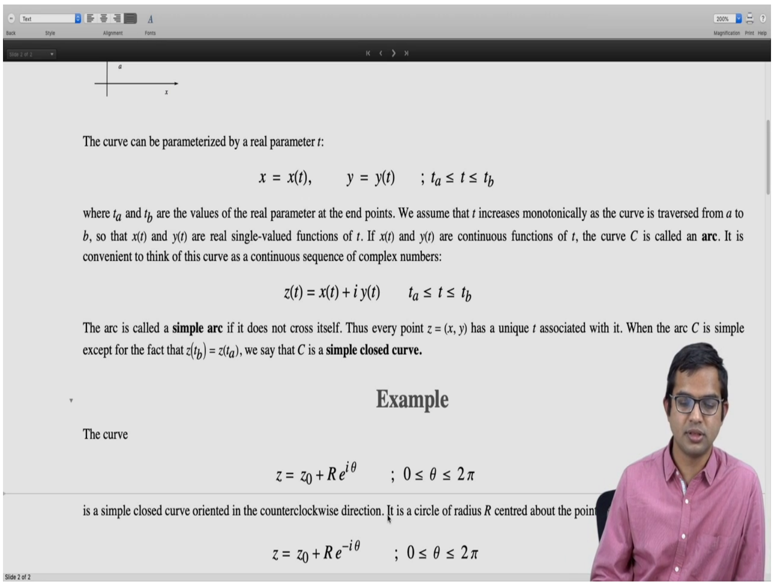

Now, this curve could be parameterized by some real parameter t. In fact, think of this parameter t as some time - the curve is generated as a function of time. You start at time t equal to 0 or t equal to t a in this case and go up to time t b and maybe you know measure the time as you draw a curve of this kind.

So, typically we are interested in curves which have some nice properties which we will just state up front. So, ta and tb are real numbers. They are the values of the real parameter at two end points, we assume that t will increase monotonically. It's helpful to take t as time and it keeps increasing and both x of t and y of t are changing as a function of time.

So, you can think of this complex number which is made up of x and y both of these real parts and imaginary parts themselves are changing as a function of time and we will look at a scenario where both x of t and y of t are continuous functions of time right.

And so, then the curve is called an arc, it is convenient to think of you know these curves are continuous sequence of these complex numbers like I just said and this arc would be called as simple arc if it does not cross itself right.

So, we do not consider curves where there are lots of criss-crosses and so on are not convenient for the purpose of contour integrals which we will look at in some detail as we go along.

So, simple arcs are useful for us and then we make this one exception where you know if your curve comes back to where it started then it is called a simple closed curve right. So, there is just exactly one point where you know the z of t b is equal to z of t a right. There are two different t's for which you get the same z that is right at the initial point and right at the end point and such a curve would be called a simple closed curve.

So, in general every point through which your curve passes has a unique t associated with it, you cannot find two different ts at which this curve may reach the same point.

So, let's look at an example. So, if you consider a path like this z is equal to z0 plus R times e to the i theta and then you. In fact, theta is like a t here right. So, theta is changing from 0 and going all the way up to 2 pi it is a simple closed curve right you. And its basically it traces out a circle whose centre is z0 and whose radius is R and you start at theta equal to 0 and then you know you complete one circle.

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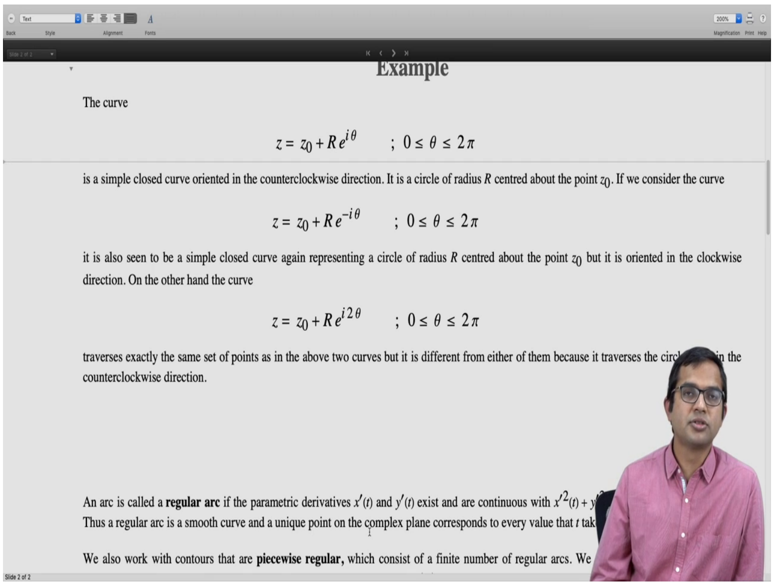

But if you consider and it is the orientation is in the clockwise direction right because 0 its going from 0 to 2 pi, but if you looked at z equal to z naught plus R times e to the minus i theta and again you let theta go from 0 to 2 pi. Now again you will get a circle, but it is in the clockwise direction right.

So, on the other hand if you consider something like z equal to z naught plus R times e to the i 2 theta and you let it run from 0 to 2 pi theta. So, then this is also a circle of radius R centred about z naught and it is in the counterclockwise sense, but this curve manages to go around 2 times, it looks around twice right. So, I mean apparently the same type of curve means slightly different things depending upon you know the details right. So, this is important to keep it in mind.

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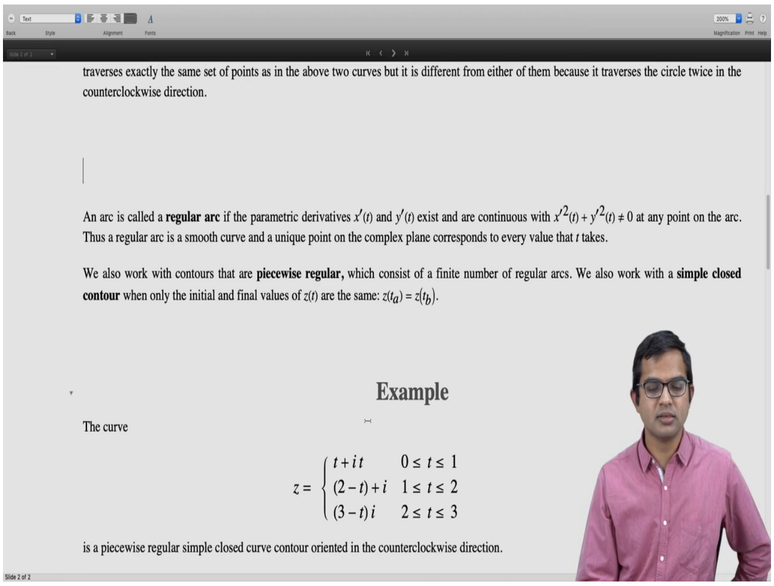

Now, an arc is called a regular arc if x of t and y of t are continuous and both x prime of t and y prime of t exist and they are continuous and also x squared x prime squared of t plus y prime squared of t is not equal to 0 at any point on the arc right.

It seems like a weird requirement. But actually if you pause for a moment what this simply means is you know the only way for the square of the sum of 2 squares to be 0 is each of them is separately 0. And such a scenario is something which we do not want to allow right.

What would that mean if both x prime of t and y prime of ts are both 0? That means, that you know there is a point at which neither x nor y is changing. So, in some sense your curve is moving along and then at some point it you know it just keeps on staying there for some time that is the type of scenario which you do not want to allow right.

So, it is there is a continuity associated with it and you know x prime and of t and y prime of t both cannot be simultaneously 0 at any point on this arc and then it is called a regular arc. There is this smoothness associated with regular arcs which is valuable when we come up with this definition of a contour integral.

So, we also work with contours which are piecewise regular right. So, there will be these sort of points where you know your curve suddenly undergoes a change and, but within every segment of the path that your contour is taking you know this regularity is maintained, but there are these in between points were you know, these conditions do not quite hold and still we can work with such curves as well. So, it is also useful to work with piecewise regular contours.

So, simple closed contour is when you know z of t a is equal to z of t b. So, the starting point and the end point are the same, closed contours are going to be very important right. So, we definitely need that notion.

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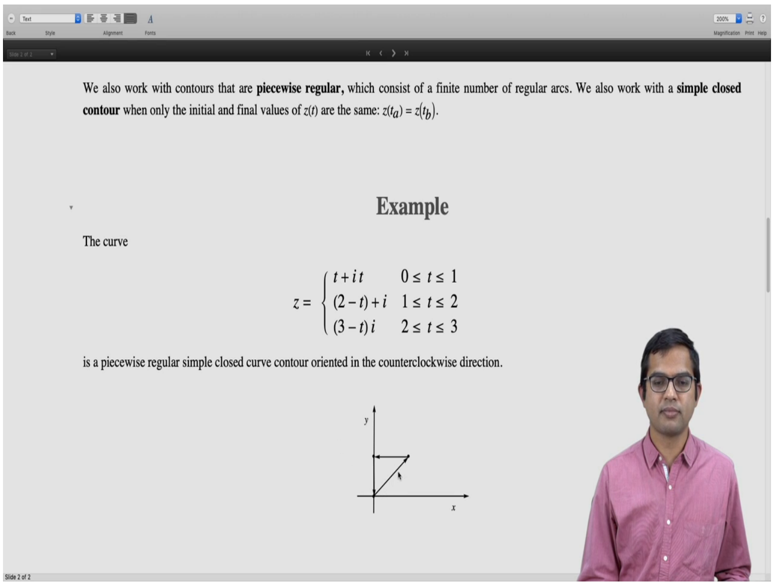

So, let us look at an example. So, if I consider the curve z is just t plus i t, when you go from t going from 0 to 1 and then 2 minus t plus i you know 1 less than or equal to t less than or equal to 2 and 3 minus t times i, 2 less than or equal to t less than or equal to 3. So, this is a piecewise regular simple closed curve right, it is oriented in the clockwise direction.

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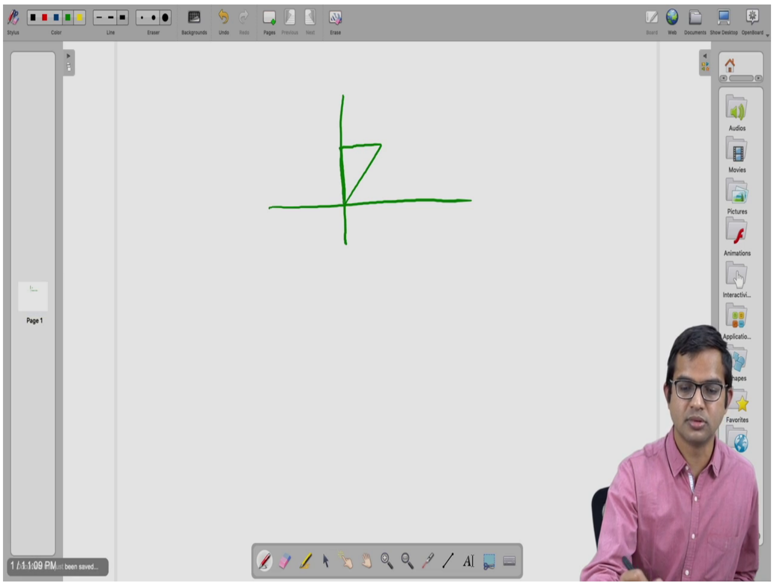

So, you can convince yourself that this is actually nothing but you know you have a, you know this is the complex plane. So, the first part of this journey looks like this and then it comes back around and then it comes back around right. So, that is what this term is which I have also traced here.

So, you start from 0 and it go up to this point along this 45 degree angle you reach 1 plus i then you come back along this direction which is parallel to the x axis and hit the y axis and then you come back down and it is a. So, it is piecewise regular we can also work with such curves.

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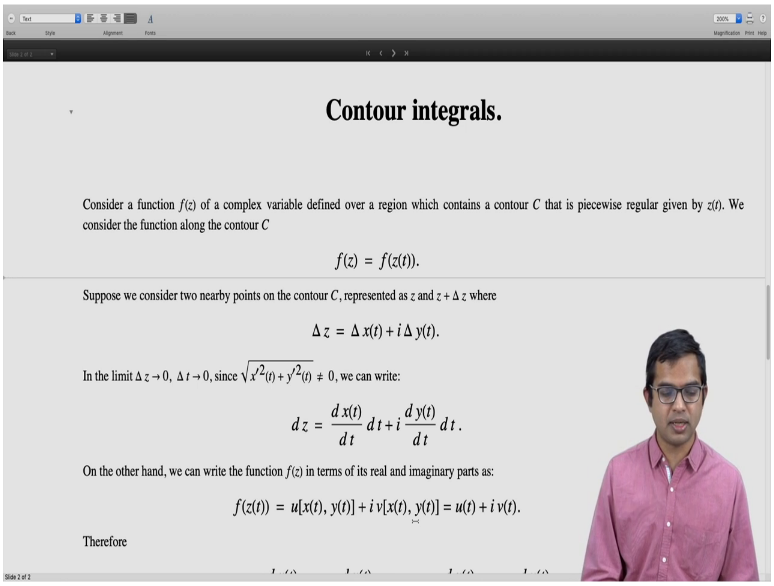

So, lets define the notion of a contour integral, suppose we consider this function f of z and a path. So, there is a contour C that is piecewise regular and it is given by z of t. So, we are looking at a function f of z of t. And then we want to see what happens, how does z change when you change t by a small amount the parameter. The real parameter changes by a small amount and so there is going to be a change in delta in x which we call delta x of t and change in y which is delta y of t.

So, as you take the limit delta t becoming very small and therefore, delta z also becomes very small, but since we have also used this condition you know x prime squared of t plus y prime squared of t is not 0 and it is a positive quantity. So, square root of this is non zero. So, what it means is this modulus of this quantity is non zero. So, we can write you know dz is dx by dt, dt plus i times dy by dt, dt.

So, basically since this quantity is non zero. So, it is meaningful to think of this dz its not going to become you know, it is an infinitesimal quantity which is not itself 0 right. So, this is you know this is an important restriction to make which we already did when we introduced the notion of a regular arc or regular contour right. I mean of course, this is violated at these very special points right.

So, suppose you have a piecewise regular contour at these points its violated, but we for the moment let us not worry about you know how to make sense at these in between points. But in general if you have a piecewise, a regular contour you know this is well defined and it works out right.

So, in some sense it is a bit like how we play with piecewise continuous functions and within functions of a real variable. So, if you have this then f of z of t is u of x of t comma y of t plus i times v of x of t comma y of t we are writing out the real part and the imaginary part separately.

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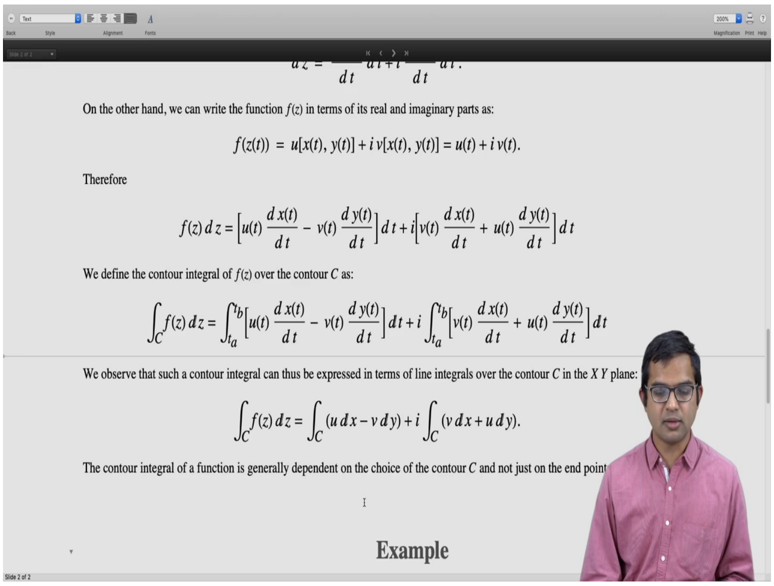

And so then we can think of this f of z of t times dz as just the multiplication of two complex numbers f of z of t itself has a real part and an imaginary part, dz also has a real part and an imaginary part.

And so you can go ahead and multiply these two and so you will get u times dx by d t minus v times dy by dt you know the whole thing multiplied by dt plus i times v of t dx by dt plus u of t d y by dt also multiplied by dt.

So, we define this contour integral. So, now, we can put in this contour integral and then take this to be an integral in terms of dt right. So, it although this is just a function of t at every point and this is just a real integral from t a to t b and. So, this information about the path that you are taking is actually embedded into this right.

So, because we have used you know this idea of what happens to z the small change in t gives you a corresponding small change in dz and that is already encoded into this. So, we see that in fact, this contour integral is a meaningful idea right if the path is you know has some reasonable restrictions that is the kind of paths we will be working with and then you have.

So, you have these two different integrals one for the real part and one for the imaginary part, which you can actually think of as two line integrals basically along these contours. So, u dx minus v dy along this contour plus i times, along the same contour v dx plus u dy.

So, basically a contour integral is made up of these two line integrals, one for the real part and one for the imaginary part and which one can write down in terms of the real part of the function and the imaginary part of the function.

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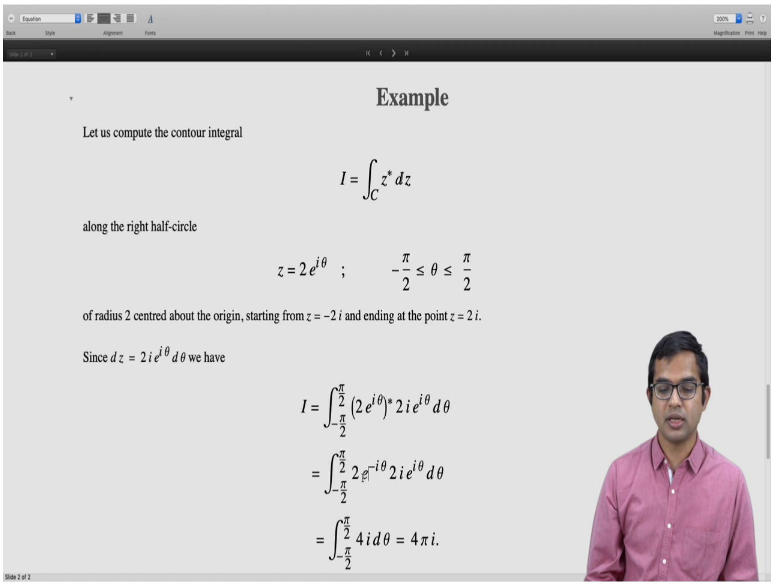

So, let us look quickly at a couple of examples of how this works out, suppose we want to compute the contour integral of z star dz along this contour which is this half, right half circle z is equal to 2 times e to the i theta right.

So, with theta going from minus pi by 2 to plus pi by 2. So, z starts from minus 2 pi and goes along you know this positive direction and reaches plus 2 i. So, it is a circle of radius 2 and with origin being the centre. So, we see that if z is 2 times e to the i theta dz is going to be 2 times i times e to the i theta d theta.

So, we have I you know is this integral from minus pi by 2 to plus pi by 2, 2 times e to the i theta the whole star we have to do and then in place of dz we have 2 times i times e to the i theta d theta, which is simply given by you know just in place of 2 times e to the i theta. I have 2 times the whole star and 2 times e to the minus i theta and then I have 2 i e to the i theta. So, this e to the i theta e to the minus i theta they anneal into each other. And so, we have 4 times i d theta which is straight forward to complete. So, and we I just get 4 pi i.

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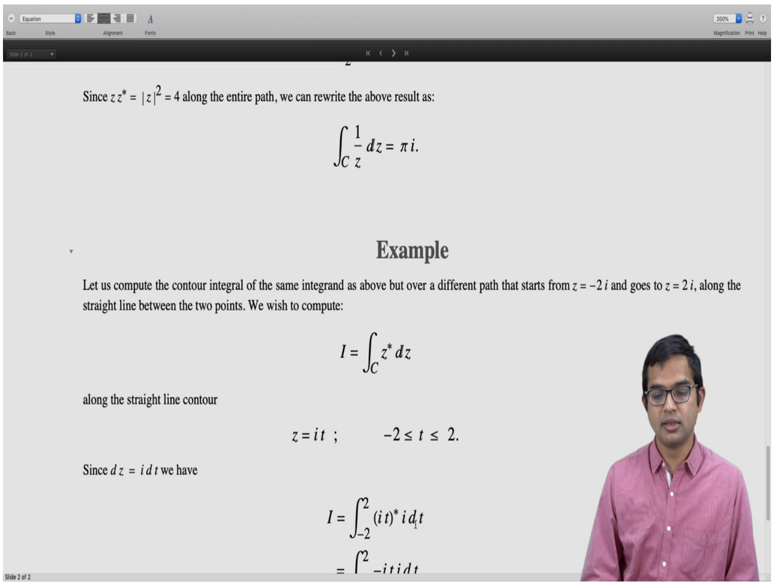

So, we also observe that along this entire path it is a circle. So, z times z star is just mod z squared which is 4 it is a circle of radius 2, so z times z star is 4. So in fact, I can think of z star as 4 over z. So, I can write this as contour integral of 4 over z dz over this contour is 4 pi i or equivalently I can say contour integral 1 over z over this path dz is the same as pi i right. So, this is completely equivalent to the result we just derived.

So, this is just you know half circle. So, we will quickly point out that if you had taken the same integrand, but over a slightly different path actually quite a different path. Suppose I started from minus 2 i and if I go to plus 2 i along a straight line. So, then I get a different answer right.

So, this contour integral in general is a path dependent operation. So, if I consider the straight line contour I take z to be i t and it starts from minus 2 and it goes all the way up to 2 t. So, initially it is minus 2 i finally, it is plus 2 i and z is i t at all at in between points and dz is i times dt.

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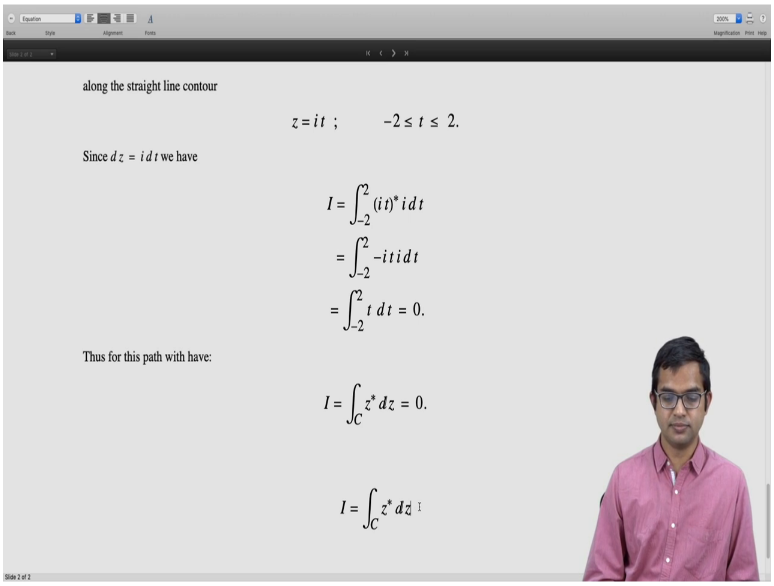

So, integral is going to be minus 2 to plus 2 i t the whole star times i dt right in place of d z i write i dt and, but I t star is. So, t is just real. So, in place of i I have to put minus i then I have 1 i here. So, minus i squared is 1. So, I get t dt; but t dt is t squared over 2 plus 2 and minus 2 does not matter it is an even function.

So, it is in fact, 0. So, thus we see that if you take this path for the same integrand from the same initial point to the same final point by, but along a different path you get a different answer, right?

So, we will come back to you know such path dependent nature of this function and sometimes you know there is a path independence as well in certain context, but this is something that we will discuss later. That is all for this lecture.

Thank you.