Mathematical Methods 2
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Complex Variables
Lecture - 23
Green's theorem

So we have been studying integral properties of functions of a complex variable. So, we will look at some very useful results as we go along, but in order to establish some of these it will be useful to recall a theorem which goes by the name of Green's theorem which perhaps we have encountered in a study of vector calculus. So, in this lecture we will discuss Green's theorem.

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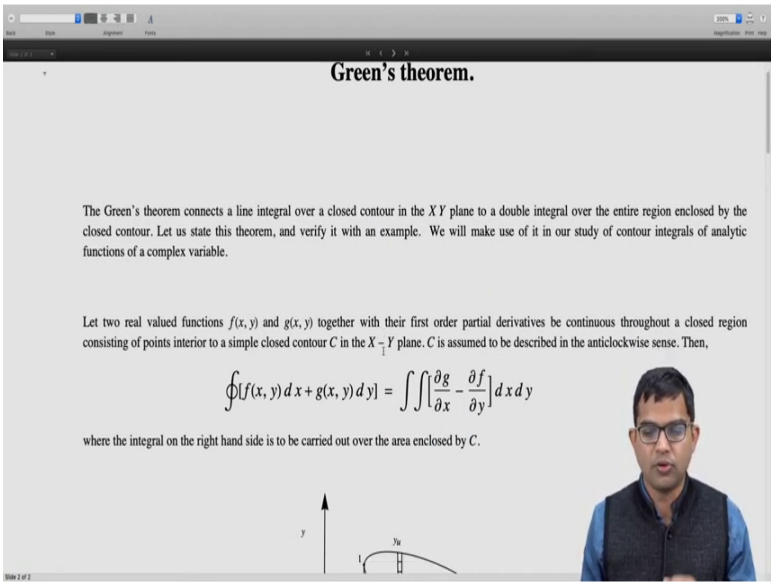

So, Green's theorem pertains to the 2 d plane and so we think of a line integral over a closed contour in the X Y plane and so, there is a way to connect it to a double integral over the entire region which is enclosed by this closed contour right. So, it is going to help us with contour integrals of functions of a complex variable. So, we will discuss it as we go along.

But in this lecture we will look at functions of two real variables real functions of two variables and so, you have two real valued functions therefore, x comma y and g of x comma y and their first order of partial derivatives are continuous throughout some closed region consisting of you know there is a simple closed contour.

And so, we are looking at all these points which are interior to this closed contour and we think of seeing in an anti-clockwise sense right if you think of it in a clockwise sense there will be a change of sign, but. So, let us say if C is you know describing in the anticlockwise sense.

Then so, the Green's theorem says that you know this line integral which is a closed along a closed path of this function f of x comma y d x plus g of x comma y d y is given by this double integral dou g by dou x minus dou f by dou y you know d x d y where this double integral is taken over the entire area which is enclosed by C.

So one can think of this as sort of a you know 2 d version of the Stokes theorem right. So, there is a little more. The Stokes theorem is a bit more general, but let us argue for Green's theorem right in this lecture and so this is a result which will be useful for us when we study contour integrals involving functions of a complex variable.

After all functions of a complex variable are some special functions of two variables x and y except that we have seen that there are some constraints which come about when you think of a function of a complex variable.

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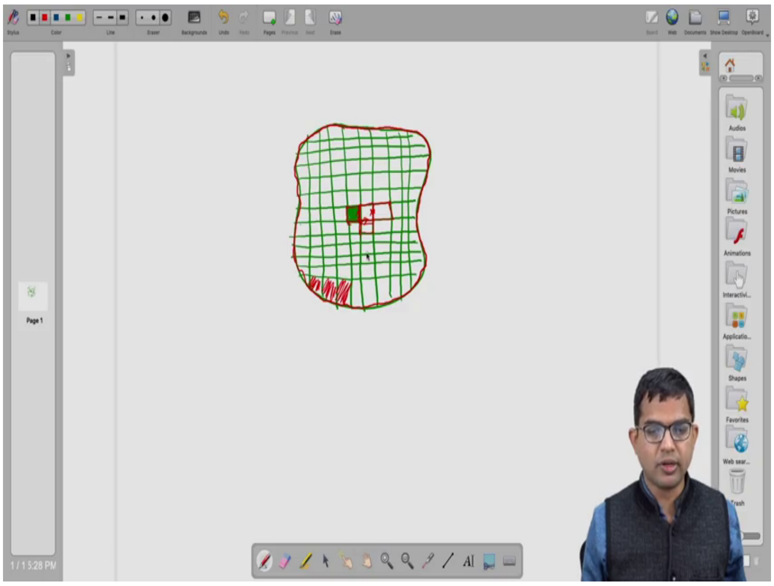

Now, you know one way to argue for this is sort of to consider a suppose you have some surface in the 2 d plane and so you break it up into lots of you know tiny regions. And then you have you know you can draw lots of small squares right each square has over a rectangle has you know dimensions of dx along the x direction and d y along the y direction.

So, let us say if you do this and you can look at how this function f of x comma y varies in some small plaquette like this right. So, then what you can do is, you can imagine taking a contour integral you know along a path like this a small tiny path like this. So, I should probably highlight it in a different color and so there is a way to argue that you know this contour integral you can show it to be. So, the equivalent of this result that we have here, but inside some small line integral and then you stitch it along with this line integral.

So when you add the line this line integral plus this line integral you will see that you know you traverse in one of these plackets along the positive sense, but along the negative sense in the other direction. So, the contribution from this part will just go away and likewise you can add this part and once again this guy is traverse in both directions and that is true in the other direction as well.

So, whatever is traversed along this direction will be cancelled by you know another component from the other direction. So, you will see that eventually all that counts is if you do this kind of an exercise carefully and if you argue that basically this contour integral will be the sum of all these small you know pieces of information that you have added from each of these small d x by d y rectangles right.

So you can go about and argue it you know in this sort of geometrical way first of all look at how f of x comma y varies in a small region as x with x becomes x plus d x and y equals to y plus d y how does you know f vary you can bring in some Taylor expansion argument right.

So, I will not go in detail into this line of argument it is something that you can look up in some textbooks or you can try to work it out yourself right, but let us look at another way of seeing this let us argue for this in the following way.

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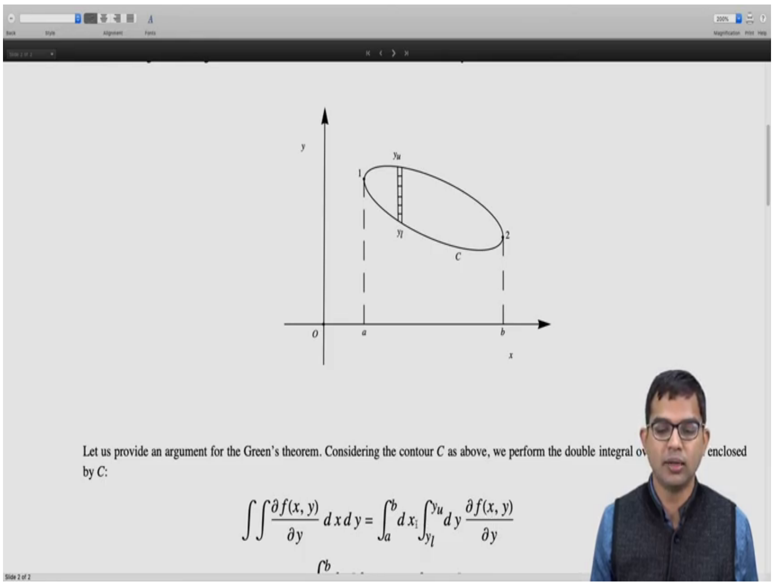

Suppose you have this contour given by you know I have represented it pictorially like here and I am thinking of going from 0.1 to 0.2 in this manner right. So I have you know this contour is made up of two parts.

So, this is what I am calling y l of x which is the lower part which is a function of x and as x goes from a to b and then you go from 2 to 1 along the upper path, so that is what I have labeled as y u. So, I can actually think of this entire thing as made up of you know strips along the vertical direction so, y l to y u and for which will depend on x.

So, if I consider this double integral of some function dou f by dou y d x d y, then I can write this as integral a to b d x right I will consider some small strip d x and then I you know x itself will go all the way from a to b. So, I have d x a to b and integral d y goes from y l to y u of course, implicitly y l and y u are both functions of x and then I have dou f by dou y.

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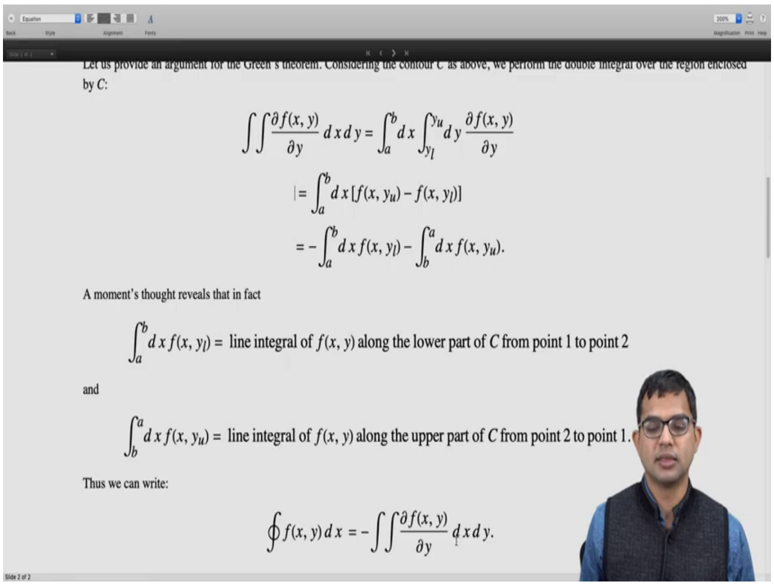

So, then immediately I can integrate you know the second one out. So, I have integral a to b d x f of x comma y u minus f of x comma y l right. So, x comma y u is you know from a to b I can also you know rewrite this as minus a to b x. So, I bring in the second term first and there is a minus sign. So, minus a to b d x f of x comma y l.

And then instead of just writing it as you know from a to b I write it from b to a and for the upper curve right. So, the reason I do that is because it makes sense to go from b to a when I am considering the upper curve so, f of x comma y u right. So, if you go from b to a; that means, you are traversing along this direction.

So the minus sign is something that we can deal with, but basically you know geometrically we can see that really this whole thing you know if you pull out the minus sign outside is basically you know this line integral from a to b along the lower path plus the line integral from b to a along the upper path which is nothing, but the closed contour integral right it is a closed line integral of this function f of x comma y right.

So, there is an overall minus sign as well. So, what we have seen is so, in other words integral a to b d x f of x comma y l is really a line integral of f of x comma y along the lower part of C from 1 to 2 and integral b to a d x f of x comma y u is line integral of f of x comma y along the upper part of C from 0.2 to 1.

So if I use these two results immediately I see that this closed contour integral f of x comma y d x is actually nothing, but minus the double integral dou f by dou y d x d y right.

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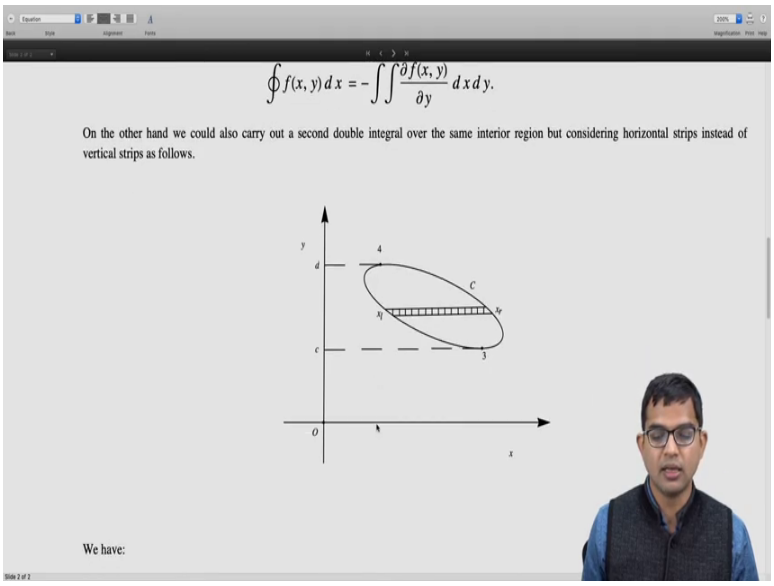

And we could have done a similar kind of thing, but along the you know considering strips along the other direction right so, vertical instead of vertical strips if I consider horizontal strips like here and then I consider a different function. So, now, I mean I still have a anti clockwise sense. So, I go along from 3 to 4 along the right direction I call this x r and then 4 to 3 now this is x left x l.

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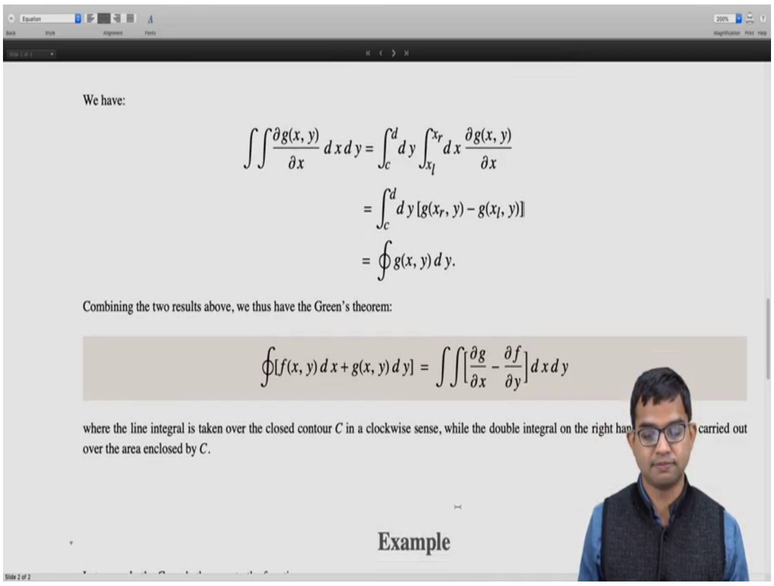

And so, now, if I consider this double integral of function dou g by dou x d x d y within this entire region which is enclosed by C then I have you know integral from C to d d y integral x l to x r d x dou g by dou x right. And then integral over if d x dou g by dou x is something which we can evaluate immediately.

So, we have integral C to d d y g of x r comma y minus g of x l comma y which is immediately seen to be a you know this closed line integral right. So, g of x r comma y is this guy you go from 3 to 4 and then you have a minus g of x l comma y going from C to d which is the same as plus g of x l comma y going from 4 to 3 right. So, that is the same as this closed line integral.

So, immediately if I combine both of these results we can go ahead and write this closed line integral of f of x comma y d x plus g of x comma y d y along this closed contour in the clock in the anti-clockwise sense is the same as dou g by dou x minus dou f by dou y d x d y which is really nothing, but the Green's theorem right.

So, this is something that you could have also worked out by the sort of geometrical argument that I gave you right. So, think of a you know function and how it varies as you change you know f of x comma y to f of x plus delta x comma y in the vicinity of the point x comma y you know you change x to x plus delta x and y to y plus delta x and then you observe how it changes and then you try to perform this well what is a double integral here will be just a small area.

Consider the area of that small plaquette that I showed you and then there is a way to show that that will be the same as the line integral and then you stitch together all of this in this geometrical fashion and then that is going to be the line integer. So, that is another alternate way of seeing this right.

So, this is Green's theorem and yeah it is important to emphasize that in this the line integral is taken in the clockwise sense and the double integral here is carried out over the area enclosed by the closed contour C.

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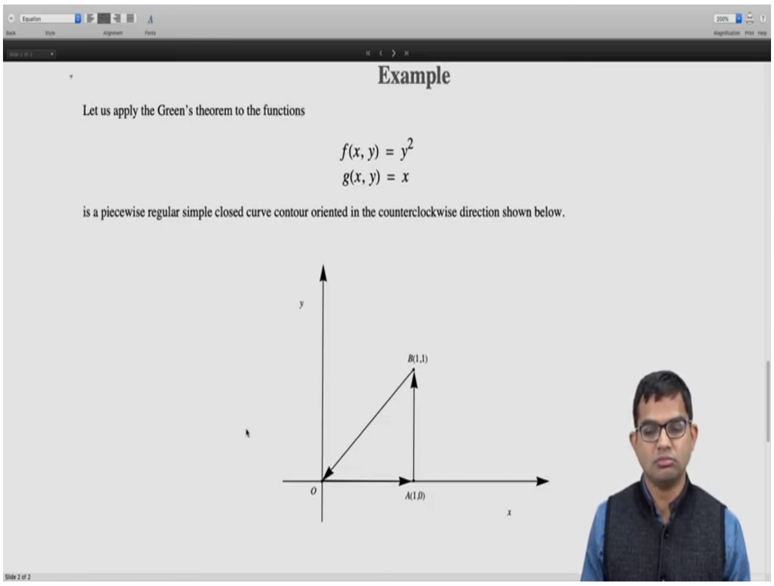

Let us look at an example of how this plays out. So, suppose you consider the function f of x comma y equal to y square and g of x comma y is equal to x and so if you know consider some piecewise regular simple closed curve contour which is you know shown like here suppose you consider this to be your contour.

So, you will start from the origin go to the point A go to the point B and come back to the point o right. So, this is a piecewise simple closed curve right. So, if you do it if you consider the some contour like this you can try out some other and verify it as well.

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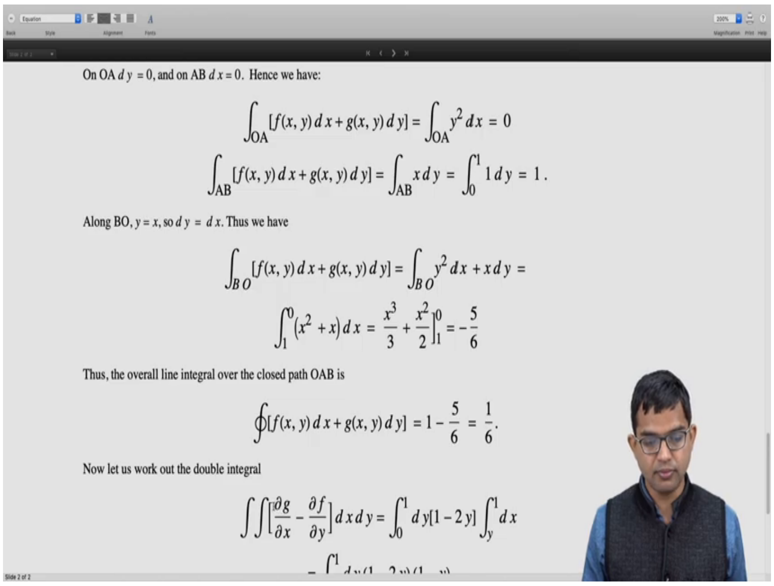

So, now we will see that it is actually quite straightforward to verify this. So, on OA d y is equal to 0 y is not changing it is only x that changes and on AB d x is 0, x is constant and it is equal to 1 and you know it is only y that is changing.

So hence we have if you look at OA integral f of x comma y d x plus g of x comma d y is nothing, but actually you know since d y is 0 it is just f of x comma y d x and, but f of x comma y is y square. So, y square d x, but y itself is 0. So, it is actually nothing, but 0 for o A. So, the o A part gives you nothing, but 0 for this these set of functions.

Now, but if you look at A to B f of x comma y d x plus g of x comma y d y now we have already said that d x is 0 on the line A B, but d y of course, is not 0 and then you have to put in x, but x is a constant. So, we are in place of g of x comma y we have plugged in x, but x is a constant on this line and x is equal to 1 in fact, so you get integral 0 to 1 d y which is nothing, but 1 right.

So there is only this path b to o where we have to evaluate it. So, on this curve we see that y is equal to x because it is you know from 1 to 1 to the origin 1 comma 1 to the origin and so, it is just the straight line which is at an angle of 45 degrees to the x axis. So, thus we have integral over B o of f of x comma y d x plus g of x comma y d y is nothing, but now we have to do both the functions will play out y square d x plus x d y.

But y squared is nothing, but x square because y is equal to x along this path x is to goes from 1 to 0 and x is just x in place of d y we put in d x. So, both of these terms can be added. So, we have x squared plus x times d x which is just x cube by 3 plus x squared by 2 from 1 to 0 which is so at 0 there is nothing and so it is basically minus one - third minus one half which is minus 5 6 5 over 6.

So that is the overall line integral over this closed path to 0 or O A B is 1 minus 5 by 6 which is 1 over 6. So, we have to add the contributions from each of these three you know pieces and so we get 1 over 6 for the line integral.

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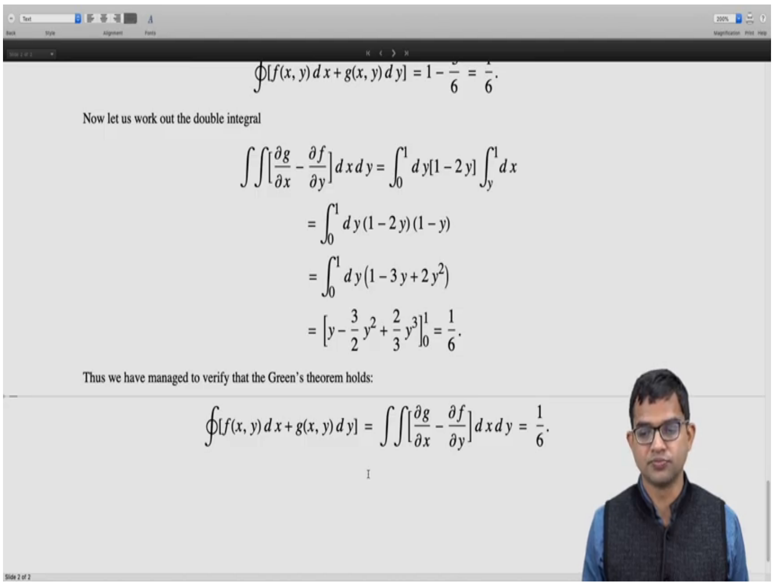

So, on the other hand let us work out the double integral. So, the double integral is straight forward. So, dou g by dou x is nothing, but so g of x comma y is x. So, dou g by dou x is just 1 and on the other hand dou f by dou y. So, f is f is y squared dou f by dou y is 2 y. So, dou f by dou y is 2 y. So, 1 minus 2 y and then we have to. So, d y times 1 minus 2 y and then we have we also have this integral d x which goes from y to 1 right.

So, you can fix the limits of integration and immediately work out. So, dou g by dou x is 1 and dou f by dou y is y 2 y and then to ensure that you are in this region you fix. So, you know d x can go from y to 1. So, d x can go from you know from y to 1 from here to 1 x, x will go from x equal to y to x equal to 1 and then y will go from 0 to 1 right.

So, this is one way of you know identifying the strict you could have also done it in the other direction, you can convince yourself that you will get the same answer right. So if you do it carefully. So, this integral this double integral is nothing, but 1 minus 2 y into 1 minus 1. So, d x will become x which is 1 minus y and then you expand you get 1 minus 3 y plus 2 y squared which is y minus 3 by 2 y squared plus 2 by 3 y cube from 0 to 1 which is just 1 over 6.

So, immediately we see that indeed this double integral is equal to the closed line integral specified according to the Green's theorem and you need both of these are equal to 1 by 6. So, indeed we have managed to verify that the Green's theorem holds for this particular case ok. So, that is all for this lecture.

Thank you.