Mathematical Methods 2
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Complex Variables
Lecture - 24
Are contour integrals path dependent?

So, we have seen how to compute contour integrals. We have defined formally the idea of a contour integral for a function of a complex variable and we saw some examples where; so, in general you know these contour integrals would depend on the path that is taken, right. So, if you are looking at some integral f of z dz and if you are going from a point A to B the precise path that you take is going to have an effect on the value of such an integral.

Now, in this lecture we look at a few examples and see is this a generic feature or are there certain special circumstances in which its only the starting point and the ending point that counts here and the details of the path are not so important. So, let us look at a few examples in this lecture, ok.

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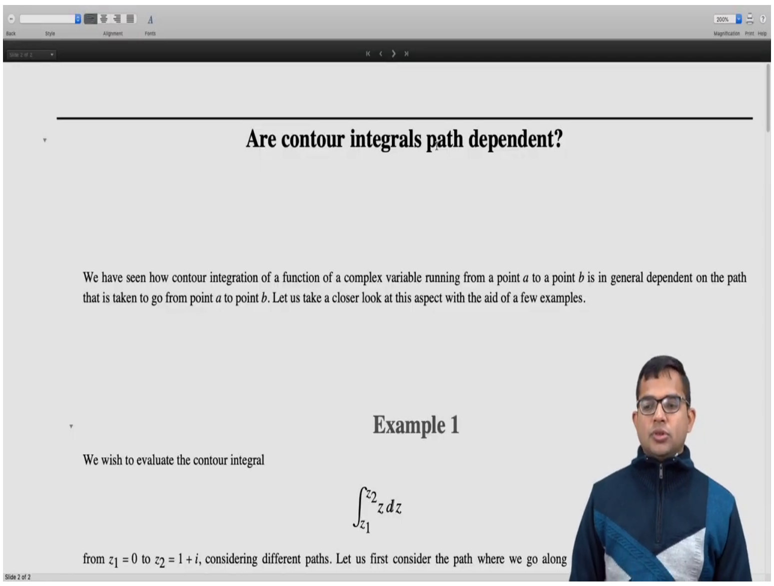

So, let us start with an example where we have the very simple degrand, it is just z. So, if you take the integrand z, and I wish to do a contour integral of this function f of z is equal to z from a point z 1 to z 2. So, I will fix z 1 to be just the origin and z 2 to be 1 plus i. And I will evaluate this contour integral considering you know a few different paths and we will see whether the value is going to be different, ok. First path that I wish to consider is represented here pictorially.

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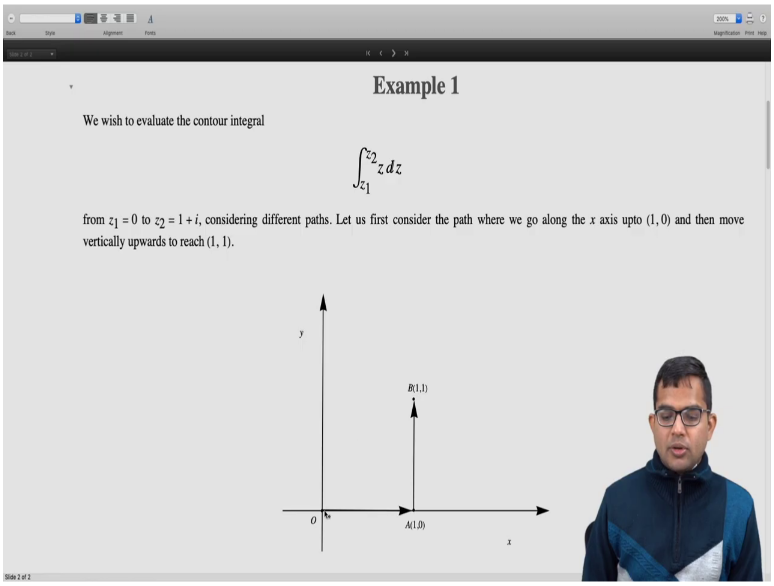

So, I need to go from the origin to the point B which is 1 comma 1, right in the xy plane. So, I want to go from 0 to 1 plus i. So, suppose I take the path along the x axis, along the real axis from 0 to 1, and then and then I turn you know in a direction perpendicular to the x axis upwards, and then I go along this path A to B and cover a distance of 1 along this direction. So that is going to take land me at the final destination, right.

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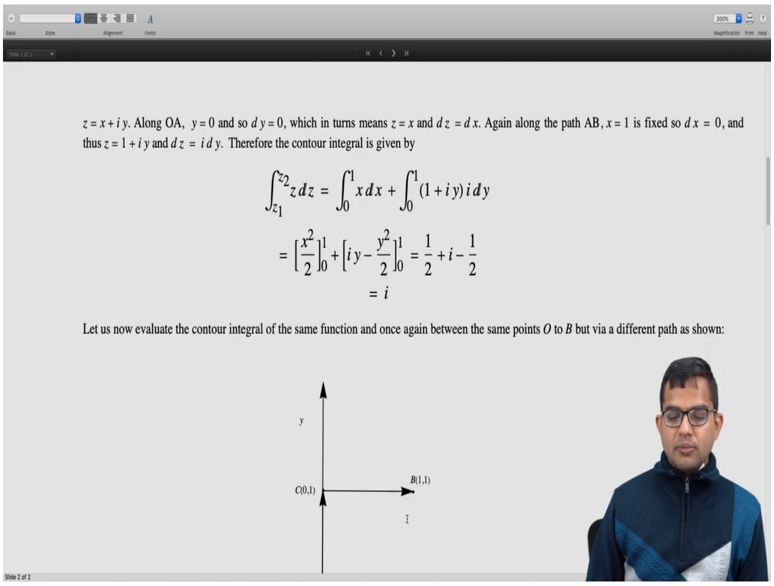

So, if I break my contour into these two sort of piecewise you know nice paths, and evaluate this integral you know along these two paths, then I have you know considering OA. So, I see that y is a constant as far as this path O to A is concerned and y is a 0, so dy is equal to 0. If z, if I write z as x plus i y and y is equal to 0, so basically z is equal to x as far as the point the path 0 to A is concerned which means I can replace z by x and I can replace dz by dx and x of course, goes from 0 to 1.

So, but on the other hand, the for the piece A to B, it is x that is a constant and its constant held fixed at 1, x equal to 1. And so, dx of course, 0 and z in this case would be you know in place of x plus i y, I can write it as 1 plus i y because x is equal to 1 and dz is just i dy, right.

So, you have to be careful about dz and da you know for different paths. And in this case there is an i times dy, right. So, I have to carefully work this out, and therefore, the contour integral can be thought of these two different regular integrals.

So, the first stretch is going to be just integral 0 to 1 in place of z, I just put x. So, in place of dz, I just put dx. And for the second path y goes from 0 to 1, in place of z, I write down 1 plus i y and in place of dz, I write i dy. So, I am I can go ahead and evaluate both these integrals these definite integrals are straightforward to evaluate.

So, I get x squared by 2 from 0 to 1 which is just one-half, then plus i times in place of i times 1, when I integrate out I get i times y and then I squared becomes a minus y, so it is minus y squared by 2 and the limits are 0 to 1. So, I get you know one-half from the first term, then plus i and then a minus half.

So, if I add all of this carefully it is just i. So, this is the value of the contour integral if I take this path along 0 to A and then A to B. Well, let us evaluate the same you know integral. So, the integrand is the same and the starting point and the ending point are the same, but I take a different path.

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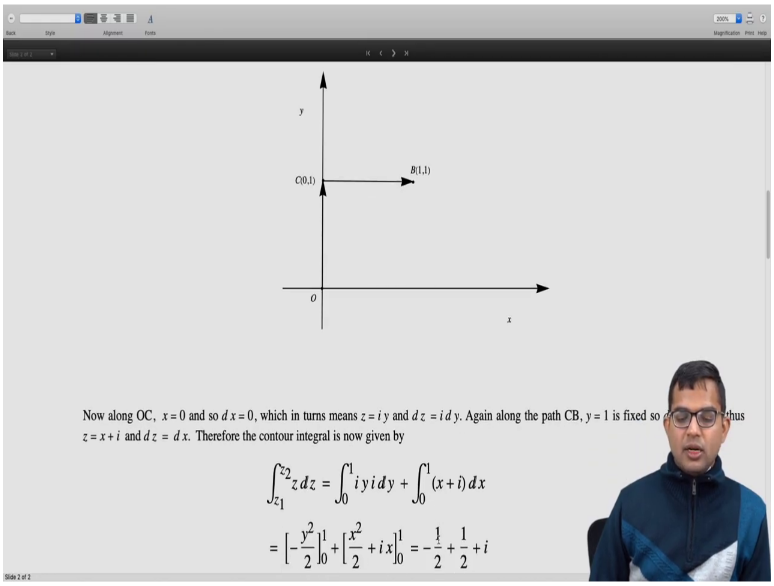

Suppose, I take this alternate path, which is first if I go along the y direction up to the point 0 comma 1, and then I take you know a right turn and go to point B in a direction parallel to the real axis. So, if I were to take this path, then once again I can argue that as far as the path 0 to C is concerned, x is a constant here and it is exactly 0 in fact.

So, therefore, z is going to be just i times y and dz is just i dy because dx is 0. And on the path C to B it is y that is a constant and it is fixed at 1. So, z is equal to 1 plus, so x plus i times y is x plus i in this case and dz is nothing but dx, right. So, this time the contour integral again is made up of you know two regular integrals which I have to sum. And here it is going to be 0 to 1, i times y, i times dy, so because i times y for this path is z and i dy is dz. And the second integral is going to be 0 to 1 x plus i times dx.

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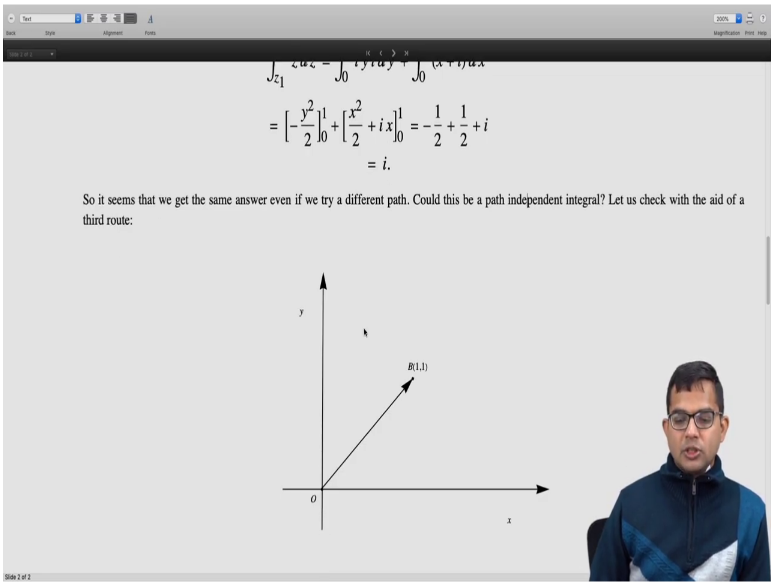

And if I go ahead and evaluate this which is straight forward twice to do, i squared is minus 1. So, I have a minus y squared over 2, 0 to 1 plus x squared by 2 plus i times x 0 to 1 which yields minus one-half plus one-half plus i and I am done. So, once again I find that the answer is i.

So, the contributions from the different stretches are different, but say they somehow add up to give me the same answer. So, now, we seem to be finding that this kind of a contour integral, for this type of integrand, seems to not care about the path; at least we have verified that for these two paths that we looked at the answer is the same. So, let us look at one more path to see if this is indeed a generic feature.

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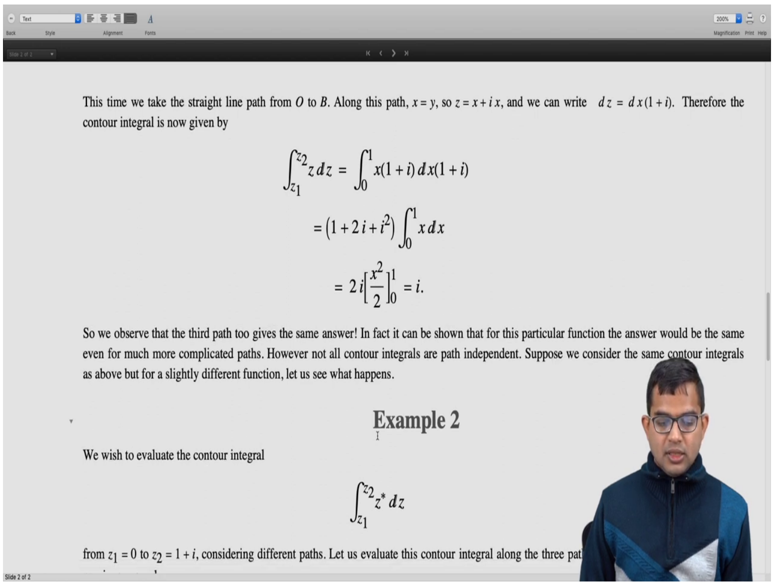

So, suppose we are going to take you know the path which is a straight line from 0 to B, if you were to change directly go from 0 to B, right. This, as the crow flies or the straight line path between 0 and B, if it took this path, then for this path x is equal to y. So, x z can be written as just x plus i x and d z is in fact, dx times 1 plus i. Therefore, the contour integral is now given by integral z dz. But z is x times 1 plus i and dx times 1 plus i is dz.

So, we just simply have this 1 plus i the whole squared which I can pull out, it is just a constant and 1 plus i whole squared is 1 plus 2 i plus i squared and then the integral is simply just 0 to 1 x dx which is straightforward to evaluate. But I notice that 1 plus i squared is 0, so it is just 2 i times x squared by 2 evaluated from 0 to 1 which gives me just a half and which cancels with the 2 outside and so, I am left with i.

So, indeed for this path also I find that the contour integral is, it gives me the same answer. So, we observe that for these 3 different paths that we tested you know the contour integral for you know this particular integrand is the same it gives me i. So, in fact, it can be shown for this particular function. It would be the same even for much more complicated paths you can try it out you can check this on your own. But there is a way to argue for this from the more sort of general perspective, right. So, we will go to that a little bit later.

But let us try and carry out the same type of example, but for a slightly different integrand suppose that is the content of example 2. Suppose, instead of integrating z dz, suppose we were to do this integral of z star dz and go from z 1 to z 2, but along these 3 different paths. Now, let us see what happens.

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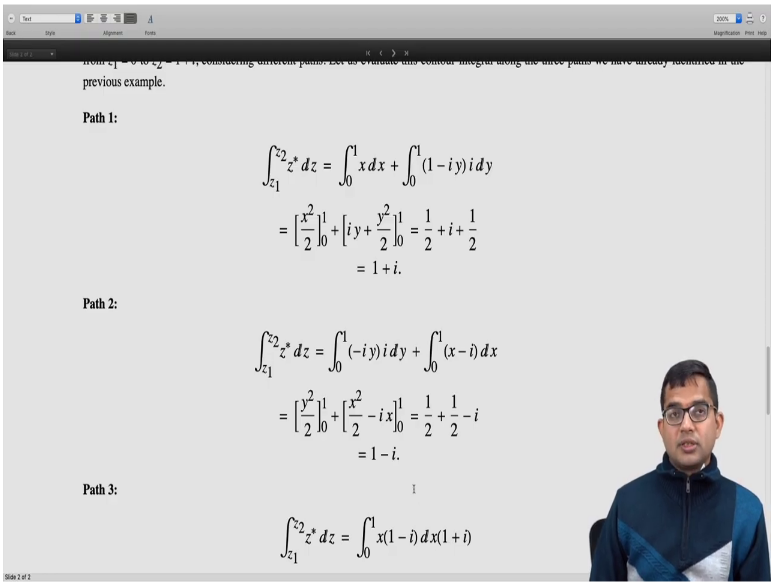

So, path 1 is when you go along the x axis first and then you take a perpendicular direction you turn left and go up. So, in this case, we get integral 0 to 1 x dx plus integral 0 to 1, 1 minus i y. So, I have to do z star and then in place of dz of course, I write i dy just like what we did in the previous example.

Now, if I evaluate this I get x squared by 2, so that gives me a half. Then, I have i y plus y squared by 2 evaluated from 0 to 1, so I get a half plus i plus half which is 1 plus i. But if I were to go along path 2, now I have you know in the direction from the in the segment 0 OC. I have z star is going to be minus i y, then I have dz is of course, i dy. Then in plus of x plus i will have to write down x minus i because I have to do z star then I have a dx.

So, if I will evaluate this carefully, so minus I squared will become 1. So, I have a y y squared by 2 and I have an x squared by 2 minus i x evaluated between 0 and 1. I get half plus half minus i which is 1 minus i. So, I immediately see that in fact, the value I get for path 1 and path 2 are different, path 1 gives me 1 plus i, path 2 gives me 1 minus i.

So, for completeness let us also quickly evaluate z star dz along the third path which is to take a straight line path. So, now, I have x times 1 minus i because its z star and then dx into 1 plus i remains as it is.

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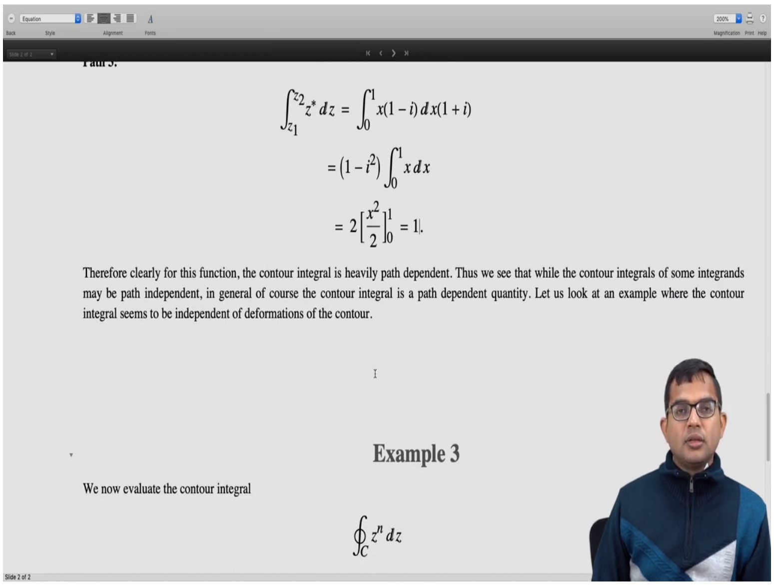

So, now its 1 minus i into 1 plus i is 1 minus i square, so that is going to be just 1 minus, minus 1 which is 2 and integral 0 to 1 x dx just gives me x squared by 2 between 0 and 1, which is just 2 times 1 by 2 which is just 1, right. So, indeed so, it is it is indeed different from both of these, you know all the 3 paths give you 3 different answers 1 plus i, 1 minus i and 1.

Therefore, clearly the contour integral as far as this integral is concerned is heavily path dependent. So, we see that, well, I mean we have already seen examples where the path does count. So, for sure the path does count as we might expect. Without doing any calculation we might think that you know contour integral of course, is defined along a certain path. So, we would expect that its value should depend on the path. But we have also seen examples where apparently it does not depend on the details of the path.

So, what are these special functions and these special paths where the path becomes irrelevant? Right. So, we will come to this, you know we will discuss this in some detail later on. But we want to look at one more example where in fact, we will see that you can also deform these contours. It is not just about going you know from a certain point to another point. So, it is a.

So, let us look at what happens when you have a closed path and you deform this path and give you an example where this deformation leaves the contour integral unchanged.

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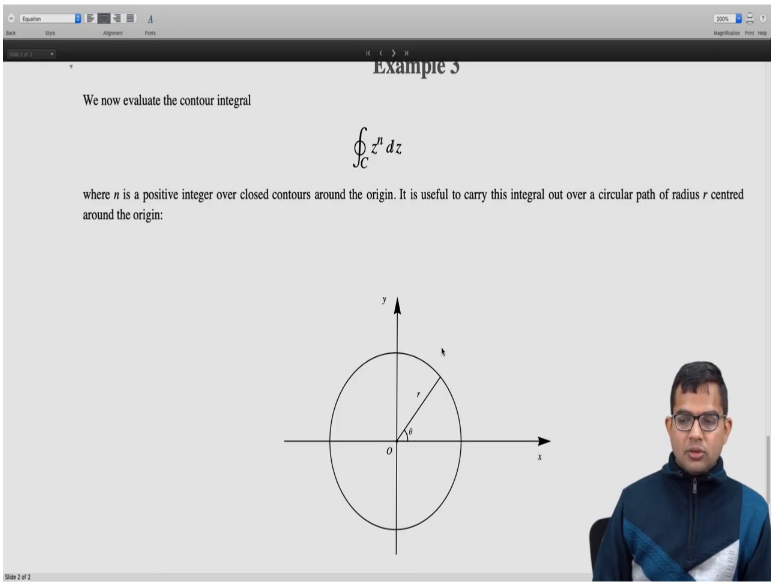

So, we have this example, where we want to evaluate integral z to the n dz and you take a closed contour which surrounds the origin, right. So, where of course, this n, it is important that n is a positive integer, right. So, if you do not have a positive integer, then the answer can be different.

So, suppose you have a positive integer and we evaluate this contour integral, and let us say we consider a circular path of radius r centered around the origin. It is easy to consider a circular path; it is convenient to carry out this integral that is why I am looking at a circular path of radius r.

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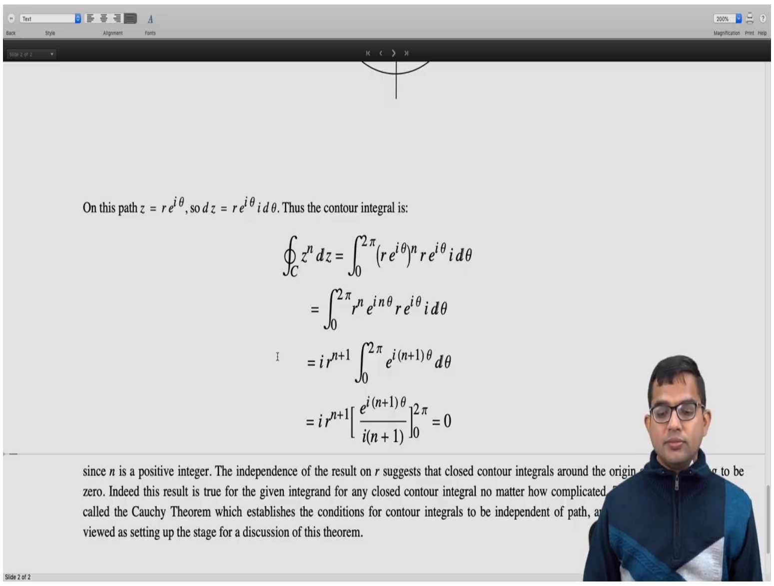

So, if I have a circular path of radius r, then I can write z as r times e to the i theta and so, immediately dz is just r times e to the i theta times i d theta. So, the contour integral can be written as just a simple definite integral, where in place of z to the n dz, I write it as 0 to 2 pi, right. So, the contour integral becomes just a definite integral in terms of theta.

So, now, r times e to the i theta the whole power n, r is not changing, only theta is the only variable which changes. So, I have r times e to the theta the whole power n times r times e to the i theta times i d theta. So, now integral 0 to 2 pi. So, r to the n, e to the i n theta, r e to the i theta, d theta.

And then I can pull out this r to the n plus 1 outside, i also comes out then it is just this integral 0 to 2 pi e to the i n plus 1 theta d theta which I can integrate out and write down explicitly the answer which is e to the i n plus 1 theta divided by i times n plus 1 going from 0 to 2 pi, which is immediately seen to be 0 because n is a positive integer.

So, n is a positive integer and so, this is going to give me the same. And theta at 0, theta at 2 pi, it is going to be the same, right. So, these things do not even do not even matter, right. So, it is all about you know just this quantity and the fact that you are going from 0 to 2 pi. So, indeed this quantity is just 0. This contour integral is, closed contour integral is a 0, if n is a positive integer.

And it is simply does not care about the radius r. So, in fact, the result is much more general for this kind of a any integrand, a closed contour integral of this kind any closed path which surrounds the origin is going to be 0 which we can show, right. It does not matter you know how complicated the path is as long; as it traverses around the origin you know in some closed path, it is going to be 0, right.

So, there is a powerful theorem which all this discussion is leading up to. And in fact, we are also going to exploit the Green's theorem which we already discussed. We sort of took a detour into the Green's theorem into a discussion of the Green's theorem. And now, we are getting back to our study of contour integrals, but with the aid of these examples we are slowly building up towards an important powerful theorem which is called the Cauchy theorem.

And the properties of analytic functions will again become crucial. So, that is coming up later on, right. So, but as far as this lecture is concerned, it is just to show you that we can find examples where there is you know path independence as far as contour integrals as are concerned. But generically speaking contour integral by its very nature is a path dependent quantity, ok.

Thank you.