Mathematical Methods 2
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Complex Variables
Lecture - 25
Antiderivatives

So, we have seen how certain contour integrals of certain integrands may be independent of the path right. And when this is true, it is useful to introduce the notion of an antiderivative right. So that is the subject matter for this lecture ok.

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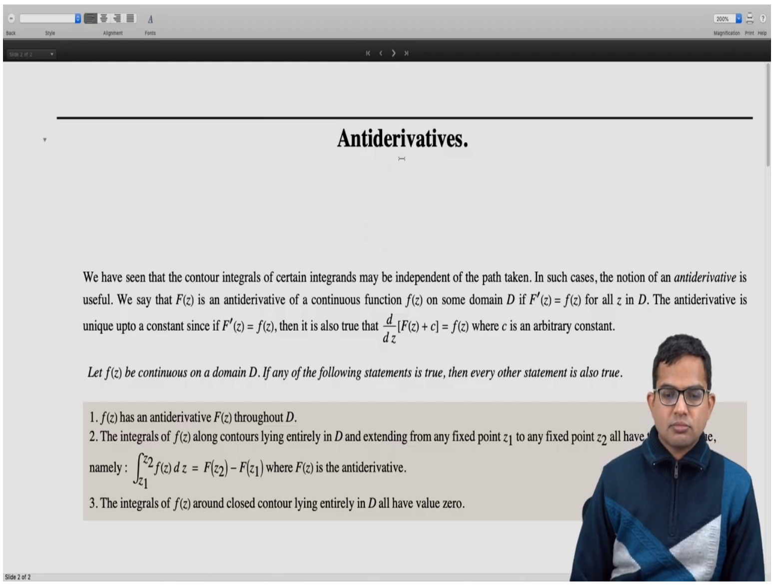

So, when we say that a certain contour integral is independent of the path. So what it effectively means is that at every point you know when you are suppose you have an integral of some function f of z that you are thinking of there is a function capital F of z that you can introduce such that the derivative of this function capital F at that point is equal to f of z right.

So, if you are able to find a function capital F of z whose derivative at any point in your domain of interest is equal to f of z – small f of z, then you call this capital F of z as an antiderivative right. And of course, this capital F of z is I mean it is unique, but only up to a constant. Because F prime of z is equal to small f of z then it also means that you can if you add some arbitrary constant to your capital F of z then that also is going to be a valid antiderivative right.

So, it turns out that you know if f of z is continuous and then if you are able to find an antiderivative which is defined throughout D right, by definition the fact that this capital F prime of z is this continuous function f of z. It automatically implies that capital F of z is analytic in the whole region and so it is just the continuity of f of z in this domain. And the existence of an antiderivative is completely equivalent to any of these three ways of looking at it right.

So, if it so happens that a function f of z has an antiderivative capital F of z throughout a domain D that is the same thing as saying that integrals of f of z along contours which all lie in entirely in this domain d and you know whichever integral you do right starting from some point z 1 to z 2, no matter which path you take as long as the entire path lies inside this domain, it is all going to have the same value.

And that value is going to be simply given by the value of this capital F function at the final point minus the value of this antiderivative at the initial point right. So, this is a bit like how we think of you know definite integrals of real variables right.

So, the statement here is that this independence the path independence of contour integrals in some domain is completely equivalent to the existence of an antiderivative, so every point f of z, every point z.

So, you have a small f of z defined, and there is a well defined capital F of z for every point inside your domain right whose derivative at that point will give you small f of z. And it is also equivalent to saying that all integrals of f of z around closed contours such that the contour lies entirely in your domain must necessarily be zero right.

So, this also is clear in the sense that if a property true holds. Then if z if you are going around some path of you know whatever the path is and then you return to where you started then it is going to be capital F of z you know z naught if you wish where you started minus capital F of z naught. So, it is just going to be 0.

So, if this holds for every part in your domain the statement is that 3 is the same as 2, and 2 is the same as 1, and 1 is the same as 3. So, any of these three statements automatically implies every other statement right. So, we will not be going to prove these statements, but it is true. And it is sort of intuitively clear that you know they are all really talking about the same thing. And then we will see some consequences of this. Let us look at a few examples where we try and apply this idea of an antiderivative.

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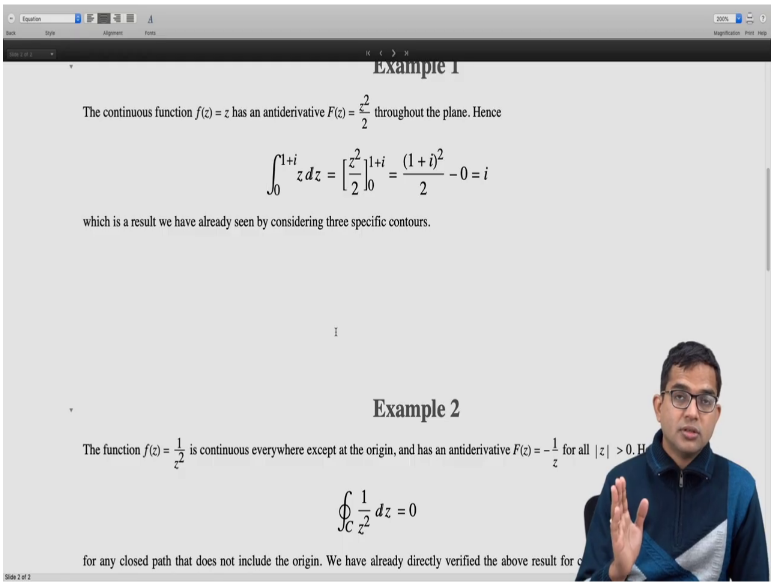

So, this function f of z is equal to z, we have already looked at this function f of z is equal to z, and it has an antiderivative. And it is clear that this antiderivative is z squared by 2 up to an arbitrary constant.

So, z square 2 by 2 because if you take the derivative of z squared by 2 you are going to get z right throughout the plane. And therefore, if you want to take the derivative from some initial point to some final point, you just need to specify the initial point and final point. If you are taking a contour integral of this kind it simply does not matter which path you are going to take.

And so the answer is simply given by z squared by 2 final point minus initial point. So, it is just 1 plus i the whole square over 2 minus 0 that is going to be 1 plus 2 i plus i square; the 1 and minus i squared will cancel. So, then you left with just i right.

But this is a contour integral this is a result which you are familiar with because we worked out the same type of a contour integral you know evaluating it over three different paths.

We first looked at you know path which went along the x-axis and perpendicular, and then we looked at one which went along the y direction, and then turned to the right. And then we also looked at a path where it went along a straight line. So, it simply does not matter which path you take because this function f of z is analytic in this entire, it is an entire function right. So, in the entire complex plane it is going to be analytic.

So, well, I mean in this case as far as this theorem that we are using is concerned we are not even really talking about analyticity of f of z right. It is simply all that is required is continuity right which is a weaker condition, but the point is that there is a well defined notion of an antiderivative, and you go through this entire path. So, it is only your final answer, your answer for a counter integral will depend only on the final point and the initial point and not on the path.

So, let us look at another example where this is true. So, suppose you take the function f of z is equal to 1 over z squared right. So, 1 over z squared is continuous everywhere except at the origin. And this has an antiderivative for all mod z greater than 0 right.

So, we can verify that minus 1 over z right it is it is analytic everywhere except at z equal to 0. So, if you take its derivative, it is going to be 1 over z squared. And so, if you take any closed contour that does not include the origin right. It should not cross the origin; it can go around the origin.

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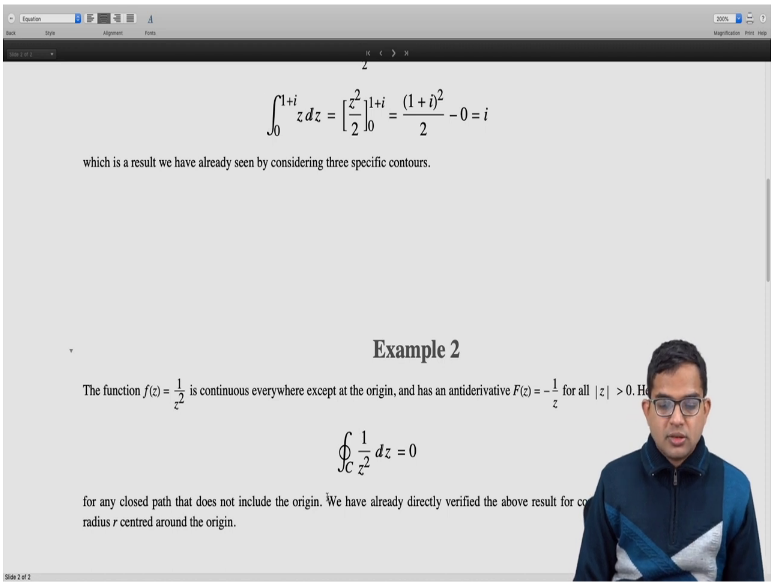

So, for this function, you are going to get 1 over z squared for any closed contour d z, it is going to be 0 right. So, we have already verified ok. So, this is actually something that we are yet to directly verify, but we, will do so by some alternate techniques right, so one way is to actually consider a circular path and then work this out.

But what we have just said is, so in this result we have looked at it from you know from the point of view of this theorem which we have not proved, but we have sort of intuitively accepted. And so as a consequence of that, we have seen that you know this contour integral 1 over z squared dz over any contour C which goes around the origin is going to be 0 right.

So, instead of saying we have already directly verified with I will say we can directly verify right. So, because I do not think we have done it yet, but it is good it is something we will look at again. So, but also it could be part of your compound.

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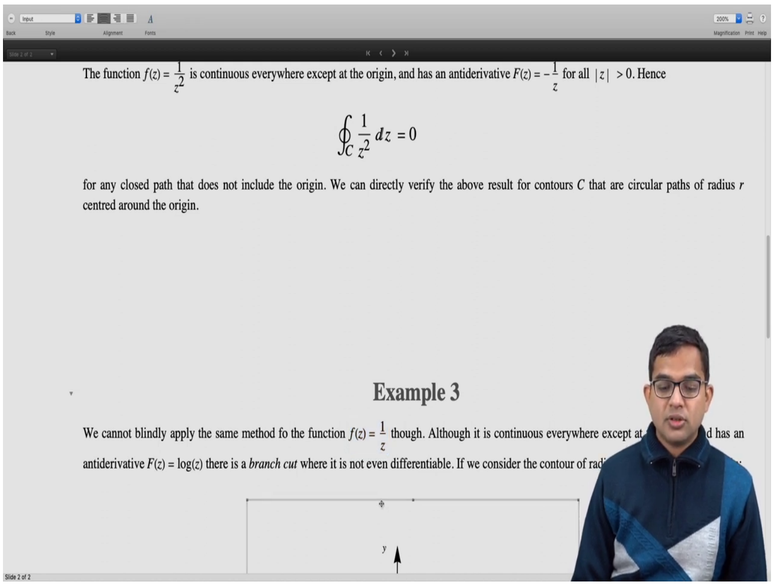

We can directly verify the above result for contour C that are circular paths right. So, what you have to do is just consider a circular path of radius r, then write down C as you know e to the r times e to the i theta and then work out this contour integral and then you can show it for circular, but paths that this result holds. But here we have just written down a more general result and that is, that this closed contour integral is 0, no matter what the shape of this contour C is.

Let us look at one more example where which looks very similar to this, but where we have to be more careful right. Suppose, we consider the function f of z is equal to 1 over z right. So, now, this function is also continuous everywhere except at the origin, but you cannot blindly conclude that you know this similar type of a contour integral for the function 1 over z dz is going to be 0, in fact, it is not going to be 0.

The reason is the antiderivative of this function 1 over z is log z which is a multivalued function, although this function f of z itself is not multivalued. So, there is no problem with this function as long as you are not at z equal to 0.

But log of z is a multivalued function and so when you are traversing a close path, you are going to necessarily cross the branch cut and that is not allowed right. So, if it is not even defined at the point you know that you are crossing, then it is definitely not differentiable.

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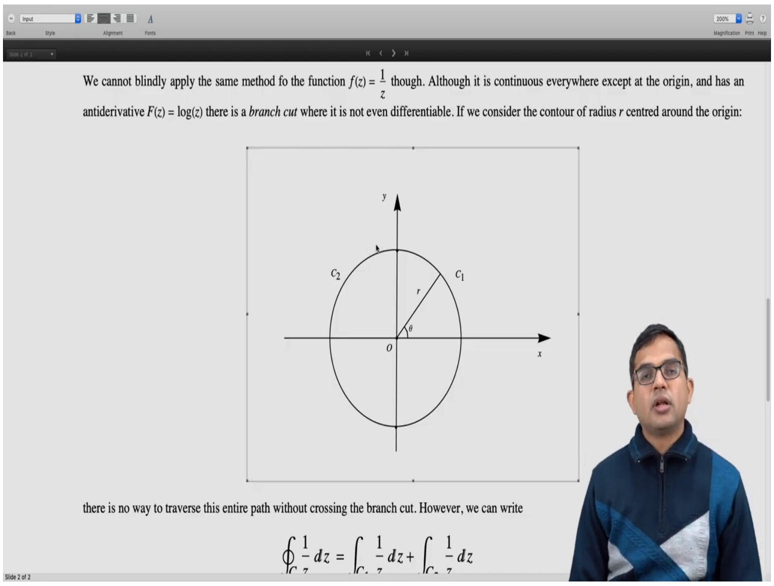

So, if we consider a contour of radius r centred or around the origin like this, so there is no way to you know complete this path without ever crossing a branch cut right. So, you cannot blindly apply this result. But what there is a trick one can do, which is to say you know if I want this entire path.

Then I can break it up into a contour C 1 which goes from you know minus r i to plus r i along C 1 and then you come back around C 2 right, so always going in anticlockwise direction. So, in fact, this contour integral over C 1 over z dz is the same as contour integral of C 1 1 over z dz plus contour integral C 2 1 over z dz.

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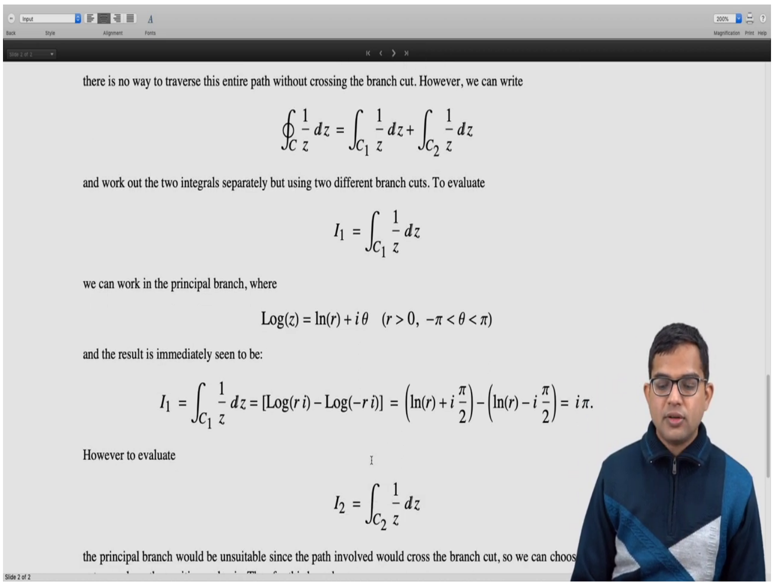

And we can work out these two integrals separately, but using two branch cuts as far as the log function is concerned. The first one is the easier one because we can directly use the principal you know branch. So, suppose I look at I 1 is equal to integral C over C 1 1 over z dz and we work in the principal branch. Then I take log z to be log r plus i theta, and r greater than 0 minus pi less than theta less than pi.

So, if I am going from this point to this point indeed my theta will always lie between minus pi and plus pi. So, there is no issue there. And so we are not crossing any branch cut if we are using this. And therefore, we can go ahead and evaluate this; you know it is it is continuous at all points. And then your resulting antiderivative itself does not ever cross any branch cut right.

And therefore, you simply get log of r i minus log of minus r i which you can evaluate to be log lan r plus i pi by 2 minus log r minus i phi by 2 corresponding to the final angle and the initial angle. And if you calculate this carefully, you just get i pi right.

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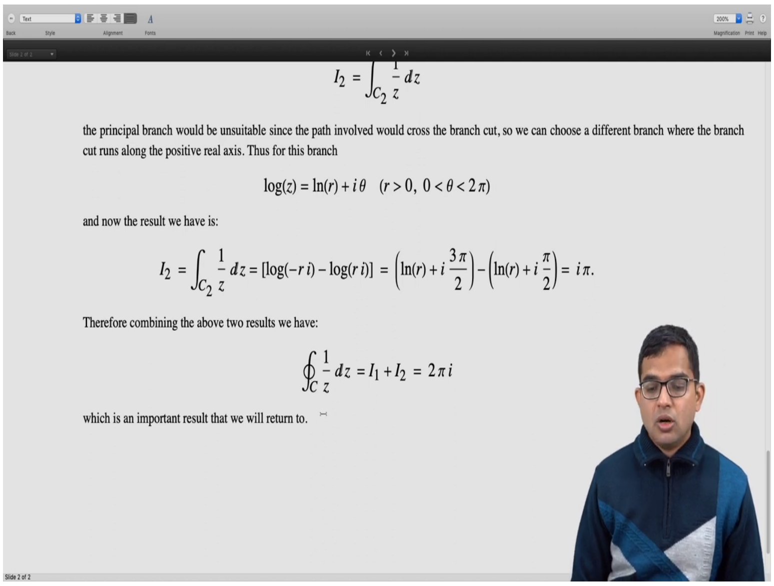

However, if you evaluate the second of these integrals over C 2, then the principal branch would be unsuitable. Because if you take the principal branch then there is a there is a branch cut along the negative axis and which you should not cross. So, as far as this integral over C 2 is concerned, we should consider a different branch.

So, let us take this branch you know take a branch in which the branch cut is along the positive axis, in other words I allow theta to run from 0 to 2 pi. If I do this, then I can go ahead and use this result which is that I take log of minus r i minus log of r i, so the final value minus the initial value.

And so in this case it will just turn out to be log of r plus i 3 pi by 2 minus log of r plus i pi by 2 which is i pi. So, I can just stitch these two together, and then I get i pi plus i pi which is 2 pi i. So, in fact, so the contour integral of the function 1 over z dz for a contour of this kind which goes around you know in some you know closed contour around the origin is not 0, but actually 1 over 2 pi i, and it is 2 pi i right. So, this is a result of very great importance.

And we will return to it later on right. And it has very important consequences and so this is the first time sort of we are explicitly stating this. You can also work this out by other methods right. So, what I thought that it is useful to point out that when we are working with antiderivatives it is important to be careful right. Although this function 1 over z itself is which is not multivalued.

So, in some sense at any point you are going along this contour, it is completely well defined, you are not doing anything illegal. But the thing is that if you want to directly use this result which is you know this path independence of the contour integral that can happen only if your you know antiderivative itself is meaningfully well-defined at all points along this contour, and you do not cross any branch cuts right.

So, that is why if you take a different branch and evaluate you know this two bits of your contour, then it is a legitimate calculation and then you get 2 pi i which is also a result which you can obtain directly from you know other methods ok. So, in this lecture, we have seen the idea of an antiderivative and how with some care we can use this notion to evaluate certain contour integrals.

Thank you.