Mathematical Methods 2
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Complex Variables
Lecture - 26
The Cauchy theorem

So, we have been setting up the scene to discuss a very important theorem, namely the Cauchy theorem. And in this lecture, we will look at how the Cauchy theorem comes about, and also look at some of its consequences, ok.

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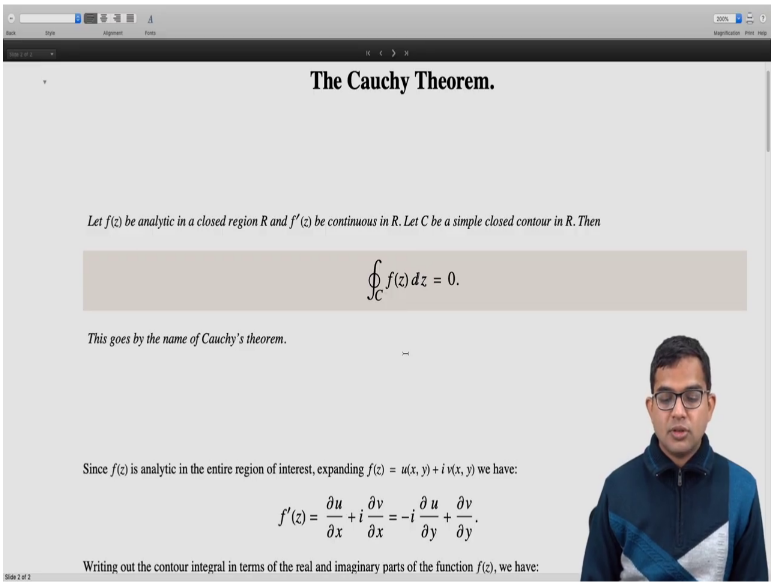

So, the Cauchy theorem is fairly straightforward to state. So, if f of z is analytic, in some closed region and f prime of z is continuous in R, right. So, let C be a simple closed contour in R, right. So, then this contour integral f of z dz is equal to 0, right. So, what is meant by a simple closed contour?

So, it is a contour which does not cross with itself. Nothing very complicated about it. It is just as simple as it gets, it is a contour which comes back to its own starting point and without crossing at any point. And this entire C must lie within your domain, right. It is a closed region R.

Now, this is the Cauchy theorem which we can actually prove, right. So, it's quite straightforward to prove when we have these conditions which goes by the name of Cauchy’s theorem. And there is a slightly modified version which we will also discuss later on, and that has a slightly different name as well.

So, since f of z is analytic in the entire region of interest, we can write you know its derivative at all these points, right. So, first of all expanding f of z in terms of the real part and the imaginary part, so u of x comma y plus i times v of x comma y, we have f prime of z is dou u by dou x plus i times dou v by dou x which is equivalently which can be written because you have Cauchy-Riemann conditions which hold as minus i times dou u by dou y plus dou dou v by dou y, right.

So, dou u. So, in place of dou v by dou x we can put minus dou u by dou y, and in place of dou u by dou x we can put dou v by dou y. So, this is just a direct consequence of Cauchy-Riemann conditions.

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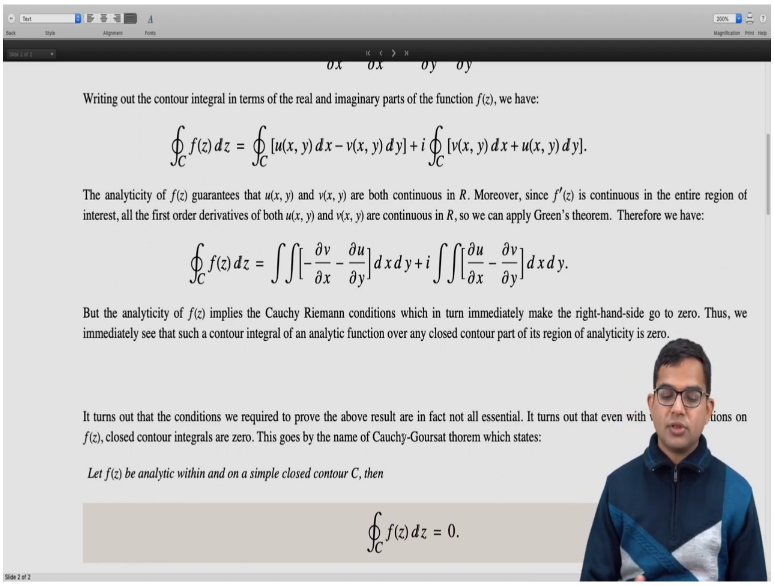

Now, if we write out the contour integral in terms of the real and imaginary parts of this function f of z, we have this contour integral f of dz is equal to contour integral over C, u of x comma y dx minus v of x comma y d y. So, this is like the real part plus i times; this another contour integral with over the same contour, but now you have v of x comma y dx plus u of x comma y dy. And this is where the Green's theorem comes in, right.

So, the analyticity of f of z guarantees that both u of x comma y and v of x comma y are both continuous in R, and moreover f prime of z is also continuous in the entire region, right. So, this is also part of the conditions which we put into the theorem. And therefore, the first order derivatives of both u of x comma y and v of x comma themselves are continuous in R. So, all the conditions which are necessary for Green's theorem hold, and then we just simply apply Green's theorem.

So, invoking Green's theorem, we see that each of these contour integrals on the right hand side above can be written as double integrals in the region of interest over R. So, now you have the first of these becomes minus dou v by dou x minus dou u by dou y, dx dy, then you have plus i times, another double integral dou u by dou x minus dou v by dou y dx d y, right.

So, but a little bit of thought reveals that, in fact, both of these terms will just go to 0 because in fact, this is nothing, but you know it looks like the Cauchy-Riemann conditions, right. So, it is like the two terms in you know one of the Cauchy Riemann conditions which says that dou u by dou y must be equal to minus dou v by dou x.

So, therefore, you know these together is just 0 and again dou u by dou x must be equal to v by dou y. So, since Cauchy-Riemann conditions hold in this entire region at every point this holds, so at every point is 0. And so, doing a double integral of basically 0 over some region. So, it is just going to be zero and therefore, immediately we see that the Cauchy theorem holds.

So, it turns out that all these conditions involving you know f prime being continuous, and all this is actually a little redundant and so, that is what the statement of the Cauchy-Goursat theorem is, right. So, that simply states that if f of z is analytic within and on some simple closed contour C, that is already sufficient for this result, right.

So, the key point is that your function f of z must be analytic everywhere inside this region and your closed contour C itself is a simple closed contour C and that is it. And so, a closed contour integral of this kind f of z dz is going to be zero.

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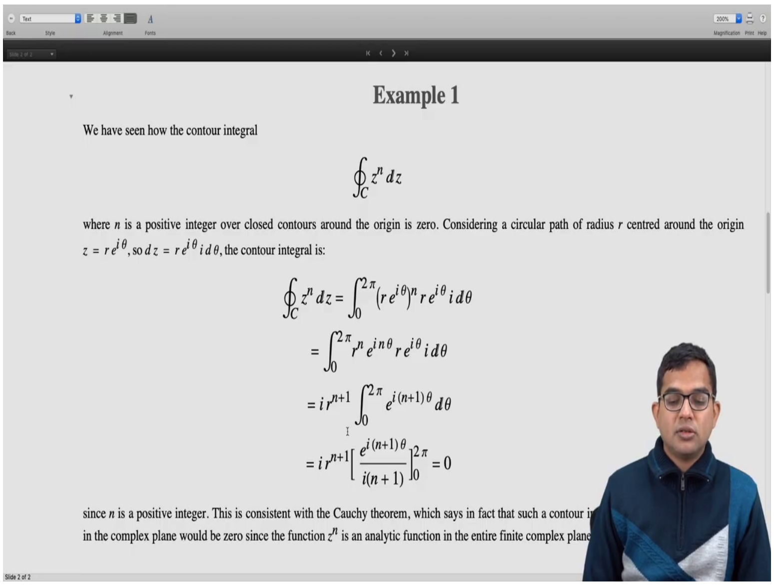

So, let us look at a few examples how this plays out. So, if you look at this integral, right. So, this we have already seen. So, if you take an integral, closed integral of the function z to the n dz, where n is a positive integer over closed contours around the origin, right. So, this is going to be 0, right.

So, consider a circular path of radius r, centred around the origin and you just put z equal to r times e to the i theta, so dz is equal to r times e to the i theta times i d theta and the contour integral is just given by this. You expand it, you know in place of z to the n you write r times e to the i theta the whole power n. You have r e to the i theta i d theta, expand everything nicely and then theta goes from 0 to 2 pi.

So, there are no difficulties, it just goes through, right. So, this is a calculation we are familiar with. And finally, we get r i times r to the n plus 1 with e to the i times n plus 1 theta divided by i times n plus 1; and 2 into evaluated at 0 and 2 pi. At both of these points it is going to be the same and you get 0, right. So, there is no I mean which is because n is a positive integer you get the same value at the ends.

And so, this is consistent with the Cauchy theorem. We need not have done any of this calculation because after all z to the n, where n is a positive integer is an entire function. So, everywhere in the finite complex plane z to the n is analytic. So, if you believe the Cauchy theorem, then automatically without doing any calculation we should be able to say that this contour integral is in fact 0, right. But, we have also explicitly checked this using this calculation.

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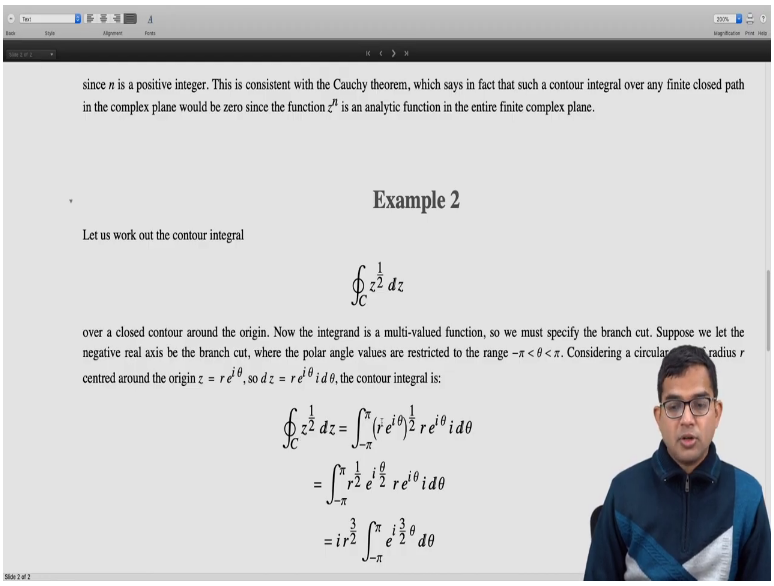

Let us look at another example. So, suppose we consider something like z to the one-half, right, in place of z to the n suppose I look at z to the half, now we have to be a bit more careful first of all z to the half is a multivalued function. So, you have to specify the branch that you are in that you are working in and specify the branch cut.

So, now, suppose we restrict the polar angle to lie between minus pi and plus pi, right. So, this is the principal branch, considering a circular path of radius r. Again just like in the previous example, now again z is equal to r times e to the i theta from which we can write dz is r times e to the i theta times i times d theta.

Now, the contour integral is you know given by this expression minus pi to plus pi, right because that is the you know restriction on the polar angle that we have put. And then we go from you know r e to the i theta to the whole power half, we can carefully collect these terms you get r to the half e to the i theta by 2 times r times e to the i theta i d theta. So, you can pull out i times r to the three-halves, then you are left with just this integral minus pi to plus pi e to the i theta 3 by 2 theta d theta.

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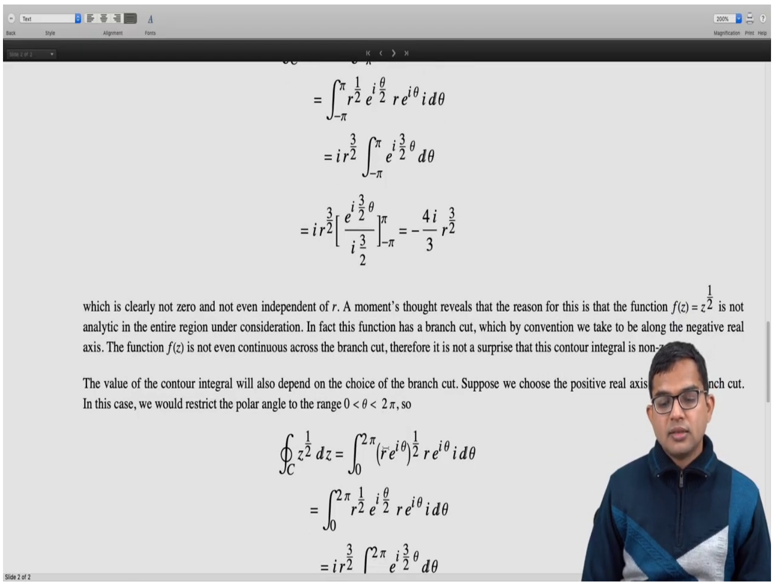

And so, if you carefully evaluate it, the answer is going to be just minus 4 by 3 times i times r to the three-half. So, we see that indeed this answer is not 0, right. First of all its not 0 and also it is dependent on r, right. So, its a path dependent quantity, right and its also I mean it depends on the size of the circle, right.

So, I mean if you think about this, why did this happen, so the reason this happens is because this function z to the half is not analytic in the entire region under consideration, right. So, Cauchy's, Cauchy-Goursat theorem does not quite hold for this type of a contour integral because z to the half has a branch cut sitting on the negative real axis.

And therefore, there is no analyticity at that point and therefore, there is no way for this kind of a you know this contour integral to satisfy the conditions essential for the Cauchy-Goursat theorem to hold. And therefore, indeed this contour integral is not going to be 0.

So, in fact, the value of this contour integral will depend on the branch in which you are working, right. So, if you are chosen a different branch cut, right and a different branch, suppose you have chosen the positive real axis to be the branch cut.

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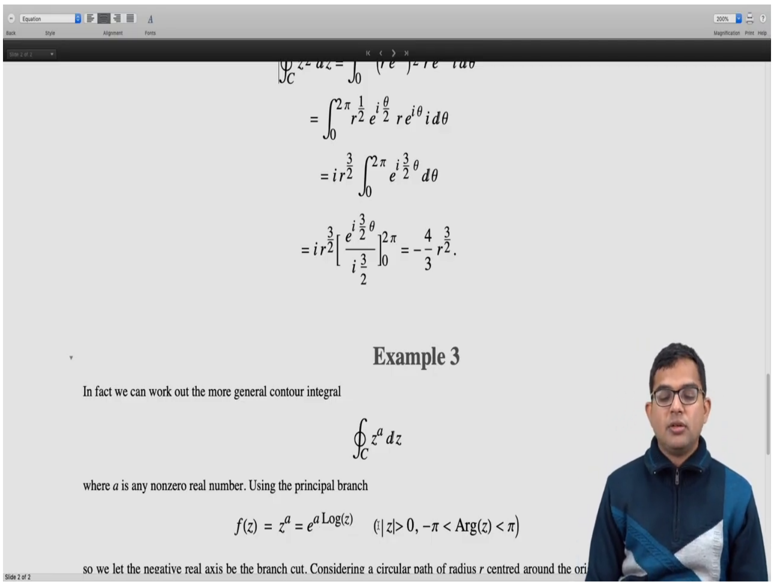

In this case, we would restrict the polar angle to lie between 0 and 2 pi. And then the answer would be you can check this will turn out to be minus 4 by 3 r to the three-half, right. So, I mean the fact that this depends on the branch you are working on is not a surprise, right.

So, this is a consequence of the fact that your integrand itself is a multivalued function, right. So, given a certain value of z you will ascribe the some value to this point z to the half, right. Since, there are many different values, so in this case there are two different values.

So, in general, if you have a multivalued function given a point z; if there are different values you can ascribe to it you know depending upon which branch you are in this kind of a contour will certainly give you different answers. So, it's not a surprise.

So, you have to pick your branch and pick your branch cut, and once you have picked your branch cut, if you want the conditions required by Cauchy-Goursat theorem to hold then you must ensure that your contour does not cross the branch cut, right. So, there are two aspects. So, this one is of course, the choice of the branch because you are working with a multivalued function and then the contour in question.

So, the contour is not allowed to cross the branch cut, if you want analyticity, right. And so, in this case since it does cross you get an answer which is not 0 and after all there is no contradiction with respect to the Cauchy-Goursat theorem. So, let us look at one more example.

So, in fact, we can work out this general result. So, instead of considering z to the you know z to the half, we can actually look at z to the a and do this kind of a similar contour integral where a is any nonzero real number, right. So, if you use the principal branch.

So, we have seen how such a function f of z can be really thought of as z to the a, which is e to the a times log z. And if you work in the principal branch of the log function which corresponds to the principal branch of the argument function, then you restrict mod z to be greater than 0 and you restrict the argument of z to lie between minus pi and plus pi.

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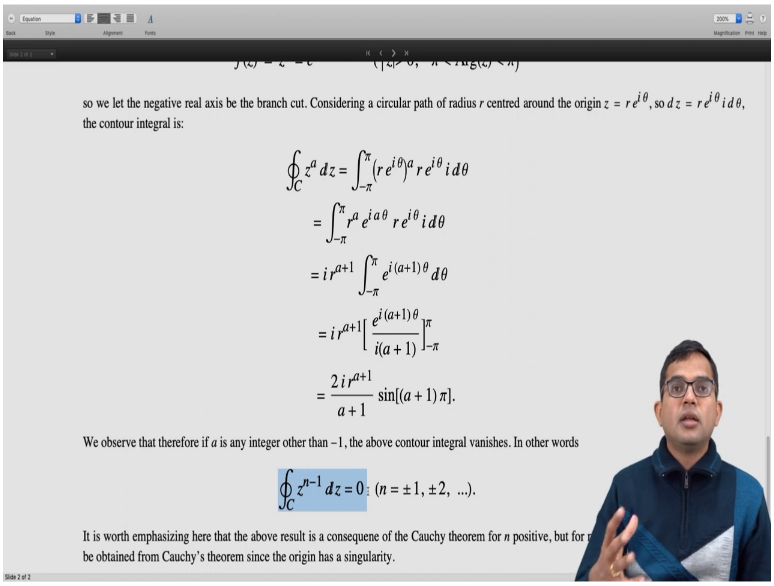

So, again we have taken the negative real axis to be the branch cut. So, now, if you consider a circular path of radius r centered around the origin, then once again you know these things are standard: z is equal to r times e to the i theta and dz is r times e to the i theta i d theta.

So, the contour integral is z to the a dz. So, you do this contour integral. Carefully collect all these sum z times r times e to the i theta to the whole power a, then it becomes r to the a times r which is r to the a plus 1, there is also an i which comes out which comes from this d theta.

Then, you are left with this integral e to the i times a plus 1 times theta, d theta needs to go from minus pi to plus pi. So, then this is nothing, but i times r to the a plus 1 is just something which you pull out.

Then, you have to evaluate this quantity e to the i times a plus 1 theta divided by i times a plus 1 between plus pi and minus pi, right. And you can write down the answer as 2 times i times r to the a plus 1 divided by a plus 1 times sin of a plus 1 times pi, right. So, some thought reveals that if a is any integer other than minus 1, then immediately we see that this contour integral actually vanishes, right.

So, in other words, we have this result: contour integral of z to the n minus 1, dz is 0 for any integer which is not 0, right. So, I have shifted this. So, instead of z to the a, I have taken it to be z to the n minus 1 and then if n is either plus or minus 1, plus or minus 2, for all integer values other than 0. So, this integral is 0.

So, this fact is of great importance and we will return to this later on. But it is worth emphasizing right away, that you know this is a you know the fact that this quantity is 0 can be extracted from the Cauchy theorem for positive values of n. But for negative values of n, you cannot use the Cauchy theorem, right.

So, the reason is because for negative values of negative integer values of n, you have a singularity sitting at the origin. So, you would have something like 1 over z squared or 1 over z cube and so on. And then it's a problem, you cannot you cannot blindly apply Cauchy theorem.

But still this integral is called contour integral is actually 0, right. So, this is something that we have worked out from first principles here. But we also saw this from the anti-derivative perspective as well, right.

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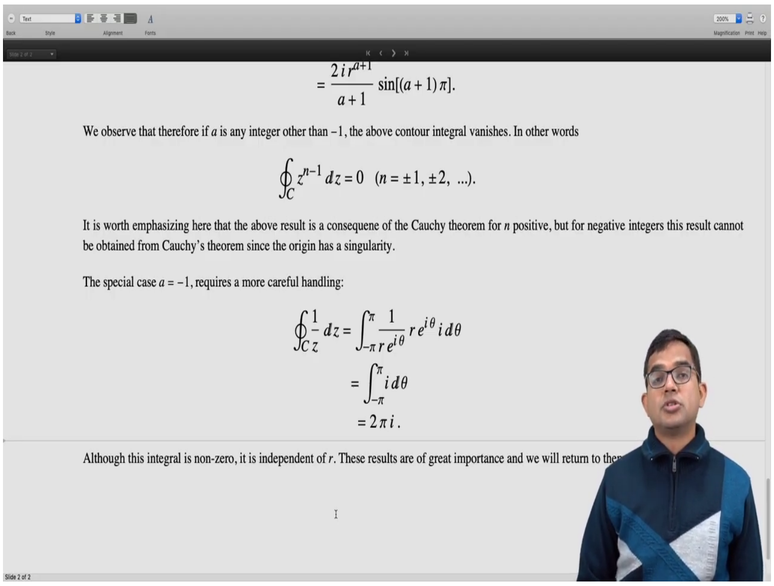

So, but when a is equal to minus 1, that is very special, we also worked out that particular integral breaking down the contour into two different parts. But here we see that you know using this general approach, we can you know we can use an approach similar to this starting with a equal to minus 1, and then we write in place of 1 over z we write 1 over r times e to the i theta.

Then, dz becomes r times e to the i theta times i times d theta. So, this r times e to the i theta actually just cancels and then you are left with just i d theta and so you get 2 pi i, right. So, this is also a valid way of approaching this problem.

And so, indeed this is the same answer which we also found by taking this you know approach where we wrote down the anti-derivative and worked out the C 1 and C 2 different portions of your contour integral. And then the final answer is 2 pi i.

So, these results of you know the contour integral of from the function z to the n minus 1, dz being 0 for all integer values positive and negative, but not 0 and also the fact that the contour integral of 1 over z dz, where the contour goes you know is encloses the origin is 2 pi i, these are you know results of great importance. And so, they have applications which we will return to at a later time, right.

So, but as far as this lecture is concerned, we looked at the Cauchy theorem, the conditions which go into go into it and how there is a you know a version of the Cauchy theorem which is called the Cauchy-Goursat theorem where the conditions are you know not as restrictive and how to make use of some of these.

We will also look at some applications of the Cauchy theorem, and how one has to be careful, and how one cannot blindly apply it, one has to check the condition carefully. So, that is all for this lecture.

Thank you.