

Mathematical Methods 2
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Complex Variables
Lecture - 27
Crossing contours and multiply connected domains

So, in this lecture we look at some generalizations of ideas we have seen. So, we have so far restricted closed contours to be simple and you know these contours do not cross themselves.

But, in this lecture, we will see how these ideas can be generalized, to allow for contours which are crossing contours when we are looking at what is called a simply connected domain. And we will also look at what happens when a multiply connected domain is introduced and how the Cauchy-Goursat theorem plays out in a multiply connected domain as well, ok.

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Crossing contours and multiply connected domains.

Non-simple contours

The Cauchy-Goursat theorem can be readily generalized to allow for contours that are not simple. Suppose we have a domain that is simply connected. A simply connected domain is one in which every simply closed contour encloses points that are in it. If our domain is the complex plane excluding the origin, then it is not simply connected. Nor is the annular region between two concentric circles. The Cauchy-Goursat theorem holds for non-simple contours in a simply connected domain. Formally,

If a function $f(z)$ is analytic throughout a simply connected domain D , then

$$\oint_C f(z) dz = 0$$

for every closed contour C lying in D .

So, what is a simple contour? A simple contour is a contour which does not basically cross itself, right. So, it so happens that the Cauchy-Goursat theorem also holds for non-simple contours provided you are in a simply connected domain.

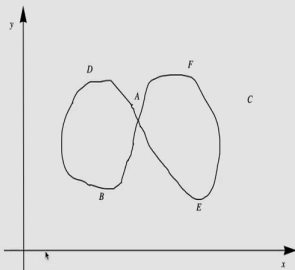
So, what is the simply connected domain? So, a simply connected domain is one in which every simply closed contour encloses points that are also in it, right. So, if you take a simple simply closed contour and look at all the points inside. Every point inside which is enclosed by a simple closed contour is also part of the domain, right.

So, in other words there are no disconnected portions in your domain. You can take any closed contour and keep on shrinking it and you will always remain within your domain. So, that is the idea of a simply connected domain, right. So, domain is like the whole region that is under consideration and contour is like you know something which you specifically consider inside that domain, right.

So, a simply connected domain is one in which every simply closed contour basically encloses points which are all necessarily inside your domain, right. So, we look at what a multiply connected domain is and then the idea of a simply connected domain will also become clear. So, if a function f of z is analytic throughout a simply connected domain, then the Cauchy-Goursat theorem holds, right; even for non-simple closed contours, right.

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Specifically we wish to emphasize that this holds for contours that cross like below:



The contour C is traversed by taking the path $ABDAEFA$. However we can actually think of the contour C as being contours C_1 and C_2 where C_1 is the part consisting of $ABDA$ while C_2 is the part consisting of $AEFA$. Thus

$$\oint_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$

So, what is a non-simple closed contour? It is something which looks like this. I have a very sort of schematic diagram of a non-simple closed contour. It is basically a contour which can cross and it can cross many times. I have drawn it to illustrate just one crossing, but you can

have more complicated non-simple closed contours. And basically Cauchy's theorem will still hold; if you are considering an integral of this kind $\oint_C f(z) dz$ over this entire path.

So, that this contour entire C can be thought of as you know; if suppose you start with A you go to the point D come down to B and then come back to A and then you go along F or you go along sorry. Suppose you start from here $A B D A$ and then you can come down $A E F A$, right. So, this is one kind of path which has to be defined carefully.

So, suppose this is what C is then you can actually think of this as being made up of 2 simply connected simple closed contours, right. So, you have a C_1 and C_2 . So, we can define C_1 to be $A B D A$ and C_2 to be $A E F A$, right. So, as far as your contour overall contour integral is concerned it is made up of a sum of you know simple contour integrals over simple closed contours. And, then the Cauchy-Goursat theorem holds individually for each of these bits.

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$$\oint_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$

But by Cauchy-Goursat theorem, each of the two terms on the right hand side separately vanish since they pertain to simple closed paths. Thus

$$\oint_C f(z) dz = 0$$

even for a non-simple contour, since such contours can always be thought of as the sum of simple closed contours.

Multiply-connected domains

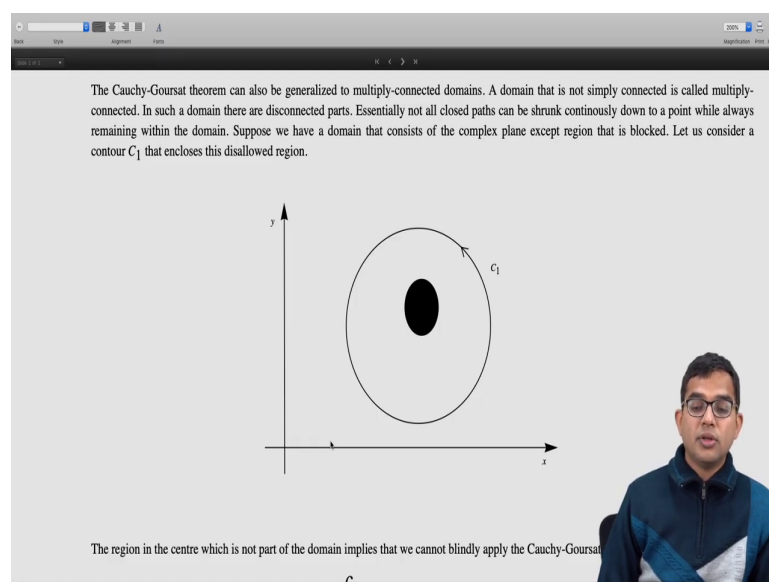
The Cauchy-Goursat theorem can also be generalized to multiply-connected domains. A domain that is not simply connected. In such a domain there are disconnected parts. Essentially not all closed paths can be shrunk continuously remaining within the domain. Suppose we have a domain that consists of the complex plane except region enclosed by a contour C_1 that encloses this disallowed region.

And so, since individually these two are 0, this overall integral; the contour integral over this non simple contour is also 0, right. So, it is fairly straightforward. You can have more complicated non-simple contours, but your overall region is simply connected, right. So, which means that there is no mess sitting in some region or in your region and then this is always possible, right.

So, the best way to understand how this is a simply connected domain is to look at what the idea of a non-simply connected domain or a multiply connected domain is. So, let us look at what a multiply connected domain is and then we will immediately be clear with the idea of what a simply connected domain is; and how is a multiply connected domain defined it is a domain which is not simply connected right.

So, let us look at an example and then you will see how a multiply connected domain basically contains disconnected parts. So, essentially not all closed paths, you cannot keep on shrinking them down to point well always remaining within the domain, right.

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So, suppose we have a situation like this. So, I am looking at a contour C_1 and everything inside this contour is part of the domain except there is some this region. So, in fact, we have an annular region between concentric circles or in this case it does not even have to be concentric circles.

So, there is some sort of mess sitting in the centre. So, there is a non-analyticity here. So, there is a; I mean for whatever reason you are considering the region which is annular to this. You are not allowing you know; you are not allowing this central region to be part of your domain and such a domain is called a multiply connected domain, right.

So, if you consider a multiply connected domain of this kind it can even be just a single point. Suppose, you take the entire complex plane, but do not allow the origin to be a part of your domain, then that is going to be not a simply connected one. Because if you consider circles around the origin and then keep on shrinking these circles in size, you will have to keep on excluding one part of it.

So, you cannot continuously shrink it you know to point to a point and allow it to be part of your domain you know. Everything inside your contour also must be part of your domain; that is not going to happen. So, it is a multiply connected domain.

Now, you cannot apply the Cauchy-Goursat theorem blindly when you have a mess sitting in the centre.

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The region in the centre which is not part of the domain implies that we cannot blindly apply the Cauchy-Goursat theorem. In particular

$$\oint_{C_1} f(z) dz \neq 0.$$

However, we can introduce bridging contours and another central contour C_2 to make the overall contour still enclose an analytic region:

The diagram shows a complex plane with x and y axes. A large outer contour C_1 is shown with points A, B, C, D, E, F, G, H. A smaller inner contour C_2 is shown with points E, F, G, H. Bridging contours connect the two contours, forming a single closed path that encloses an analytic region.

So, in particular you cannot say that this contour integral $\int_{C_1} f(z) dz$ is equal to 0, right. So, one example we have seen is that of $1/z^2 dz$ and you cannot blindly say that it is going to be 0 invoking the Cauchy-Goursat theorem, because you know there is a singularity sitting at the origin.

But, we have seen that there is some other way to argue for that particular result and that is why it is important to emphasize it. So, you know that result may hold, but it is not a consequence of the Cauchy-Goursat theorem.

Now, let us see what happens with; how can we still work with a multiply connected domain and still you know that after all this function is analytic in this entire region in this annular region. So, is there a way to come up with some other contour and still make use of the Cauchy-Goursat theorem and what consequences does that have?

So, let us introduce what are called bridging contours. So, you consider some other bit in between contours which I am calling C_2 here and where the direction of motion is now clockwise. So, you can think of this C_1 which is this outer contour and the C_2 as this inner contour.

And so, suppose we consider this entire region which lies between C_1 and C_2 . Now, that can be your you know contour of interest. And so, there of course, your function is entirely analytic in this entire region.

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Now suppose we are interested in

$$\oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz.$$

We can compute this generically because the entire region that lies between the contours C_1 and C_2 is analytic. We can see this explicitly with the aid of the bridging paths BD and GF . We can think of the sum over the two contours C_1 and C_2 be the same as over the contours $C_5 = ABDEFGA$ and $C_6 = BGFHDB$, since the paths BD and FG get traversed two times in opposite directions and their contributions cancel. Clearly the contours C_5 and C_6 correspond to simply connected regions thus, and therefore invoking the Cauchy-Goursat theorem, we have:

$$\oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz = \oint_{C_5} f(z) dz + \oint_{C_6} f(z) dz = 0.$$

Corollary: The principle of deformation of path

Let C_1 and C_2 denote positively oriented simple closed contours where C_2 is interior to C_1 .

So, you should be able to say that the sum of you know this f of z dz in C_1 plus integral f of z dz C_2 must be 0, right. In some sense you can think of this as a closed contour which you

know entirely lies in an analytic region. And so therefore, it should be 0. And, there is a way to argue for this and in a more transparent way and that is to use this bridging contour.

So, what you do is you think of this C_1 plus C_2 is basically you know you go along this. So, you can think of C_1 plus C_2 as the same as C_5 plus C_6 . We introduce these contours which we will start with A go down to B ABDEFG and A, right. So, this is one. So, this kind of a contour is one. And then C_6 is something like B BGFHDB, right.

So, you see that if I add this contour and if I add this contour B sorry B BGFHDB. So, we see that when I sum these two, I will find that in one of these cases I am moving along this direction from B to D and the other one I am going from D to B. And, likewise in one of them I am going from G to H and in the other one I am going from H F to G, G to F and F to G.

And, these parts will cancel, everywhere else if you just carefully watch what is going on it is nothing but going along C_1 in the outer contour and going along C_2 in the clockwise direction in the inner contour. So basically, and then we argue of course, for this path ABDEFGA.

So indeed the Cauchy-Goursat theorem must hold because everything inside the function is analytic; everywhere inside this region. And it is basically a you know simply connected region and it is a simple contour. So, it is going to be 0 and also in this region it is going to be 0.

So, overall this sum of these two contours is indeed 0, right. So, this is a direct consequence of the Cauchy-Goursat theorem. And, there is an important corollary from this which is also called as the principle of deformation of paths, right.

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If a function $f(z)$ is analytic in the closed region consisting of those contours and all points in between them, then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz.$$

So, if you take C_1 and C_2 to be two positively oriented, right; anti-clockwise oriented simple closed contours; C_1 is some outer one and C_2 is some interior closed contour.

And, then if your function is analytic in this closed region consisting of those contours and all points between them, right. So, we are not saying anything about this inner region. There could be a mess in there. So, what this you know the result pertaining to Cauchy-Goursat theorem for multiple multiply connected domain implies is that; the contour integral over C_1 of f of z dz is the same as the contour integral C_2 of f of z with respect to C_2 , right.

So, the reason is that you know whatever mess is there is somewhat hidden right in the centre, right which is interior to C_2 . So, then it does not matter which contour you take because you know this in the entire region between C_1 and C_2 it is analytic. On the contour C_1 and on the contour C_2 and in the entire region between C_1 and C_2 your function is analytic.

Therefore, you can actually deform contours, right as long as you do not cross any singularity, as long as you do not cross any non-analyticity. So now of course, you also notice that the sense in which C_1 moves is the same as the sense in which C_2 moves.

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If a function $f(z)$ is analytic in the closed region consisting of those contours and all points in between them, then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz.$$

Clearly this follows from the previous result since we have:

$$\oint_{C_1} f(z) dz + \oint_{-C_2} f(z) dz = 0$$

and it is simply a matter of realizing that $\oint_{-C_2} f(z) dz = -\oint_{C_2} f(z) dz$.

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And, this is a direct consequence of the previous result, right. After all the previous result says that you know if you take this and then add it to minus C_2 , right; so here you see that the sense goes in the opposite direction when you are doing it as a plus this thing. But then, you immediately argue that plus C_2 you know in this case will be you know minus C_2 , because I put it in a different direction.

So, basically what it boils down to is that you can deform contours as long as you do not cross any non-analyticity when you are doing this. So, this is also a result of great importance and we will apply it as we go along as well.

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Example

The function $f(z) = \frac{1}{z}$ is analytic everywhere except at the origin. So

$$\oint_C \frac{1}{z} dz$$

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Let us look at an example. So, we have seen that if you consider the function f of z is equal to 1 over z , it is analytic everywhere except at the origin.

And, if we consider this kind of an integral 1 over z dz over some contour C which goes around the origin, right which does not have to be a circle; any contour which is in the positive sense must be the same as you know in the same sense. If you take a different path which we can take for simplicity to be a circle of radius r .

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for any closed path that surrounds the origin should be the same as

$$\oint_{C_0} \frac{1}{z} dz$$

where C_0 is a circle of radius r such that C_0 lies entirely inside C . We have already directly computed this contour integral and found its answer to be $2\pi i$. Thus:

$$\oint_C \frac{1}{z} dz = \oint_{C_0} \frac{1}{z} dz = 2\pi i.$$

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But, we have already evaluated this contour integral of 1 over z dz over C naught and that we have found it to be $2\pi i$. So thus, we have the result that 1 over z dz over any closed contour in the positive sense which encloses the origin and does not go through the origin is the same and that is going to be $2\pi i$, right, ok. So, that is all for this lecture.

Thank you.