Mathematical Methods 2
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Complex Variables
Lecture - 27
Crossing contours and multiply connected domains

So, in this lecture we look at some generalizations of ideas we have seen. So, we have so far restricted closed contours to be simple and you know these contours do not cross themselves.

But, in this lecture, we will see how these ideas can be generalized, to allow for contours which are crossing contours when we are looking at what is called a simply connected domain. And we will also look at what happens when a multiply connected domain is introduced and how the Cauchy-Goursat theorem plays out in a multiply connected domain as well, ok.

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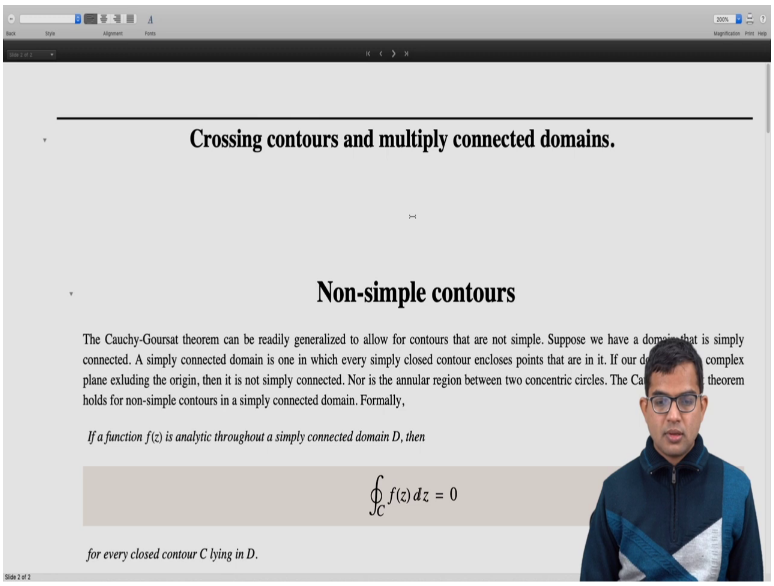

So, what is a simple contour? A simple contour is a contour which does not basically cross itself, right. So, it so happens that the Cauchy-Goursat theorem also holds for non-simple contours provided you are in a simply connected domain.

So, what is the simply connected domain? So, a simply connected domain is one in which every simply closed contour encloses points that are also in it, right. So, if you take a simple simply closed contour and look at all the points inside. Every point inside which is enclosed by a simple closed contour is also part of the domain, right.

So, in other words there are no disconnected portions in your domain. You can take any closed contour and keep on shrinking it and you will always remain within your domain. So, that is the idea of a simply connected domain, right. So, domain is like the whole region that is under consideration and contour is like you know something which you specifically consider inside that domain, right.

So, a simply connected domain is one in which every simply closed contour basically encloses points which are all necessarily inside your domain, right. So, we look at what a multiply connected domain is and then the idea of a simply connected domain will also become clear. So, if a function f of z is analytic throughout a simply connected domain, then the Cauchy-Goursat theorem holds, right; even for non-simple closed contours, right.

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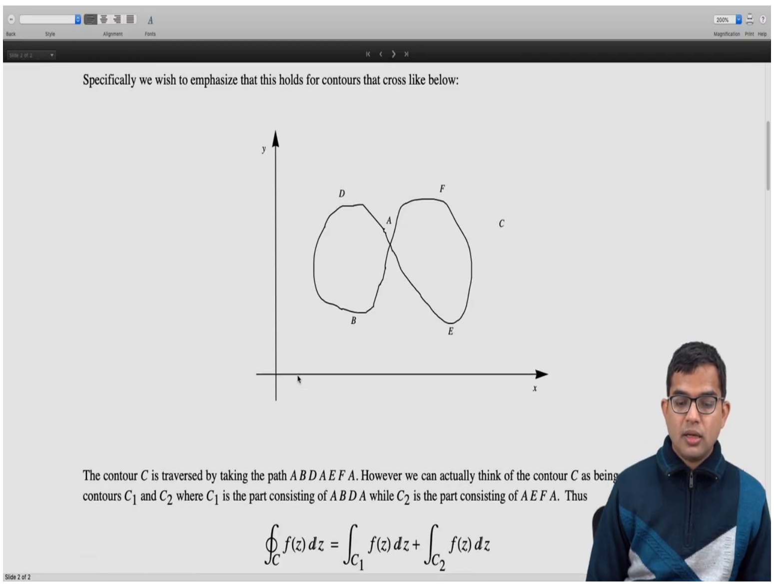

So, what is a non-simple closed contour? It is something which looks like this. I have a very sort of schematic diagram of a non-simple closed contour. It is basically a contour which can cross and it can cross many times. I have drawn it to illustrate just one crossing, but you can have more complicated non-simple closed contours. And basically Cauchy's theorem will still hold; if you are considering an integral of this kind f of z dz over this entire path.

So, that this contour entire C can be thought of as you know; if suppose you start with A you go to the point D come down to B and then come back to A and then you go along F or you go along sorry. Suppose you start from here A B D A and then you can come down A E F A, right. So, this is one kind of path which has to be defined carefully.

So, suppose this is what C is then you can actually think of this as being made up of 2 simply connected simple closed contours, right. So, you have a C 1 and C 2. So, we can define C 1 to be A B D A and C 2 to be A E F A, right. So, as far as your contour overall contour integral is concerned it is made up of a sum of you know simple contour integrals over simple closed contours. And, then the Cauchy-Goursat theorem holds individually for each of these bits.

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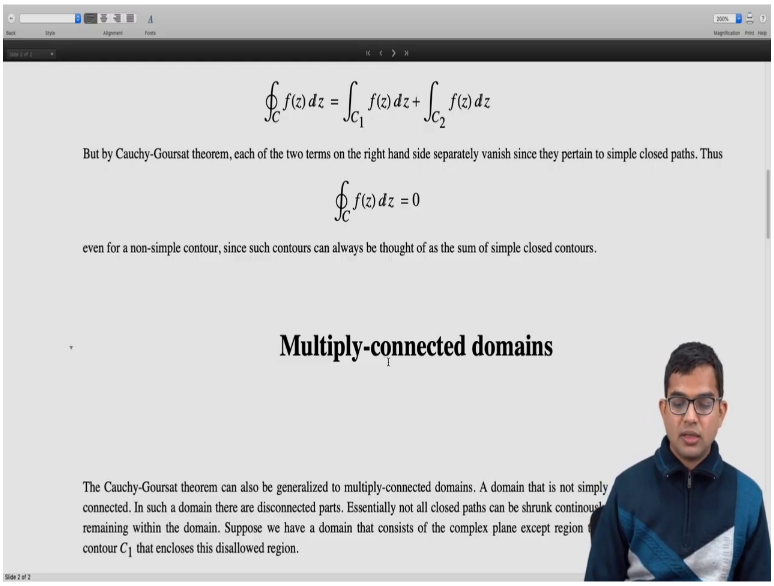

And so, since individually these two are 0, this overall integral; the contour integral over this non simple contour is also 0, right. So, it is fairly straightforward. You can have more complicated non-simple contours, but your overall region is simply connected, right. So, which means that there is no mess sitting in some region or in your region and then this is always possible, right.

So, the best way to understand how this is a simply connected domain is to look at what the idea of a non-simply connected domain or a multiply connected domain is. So, let us look at what a multiply connected domain is and then we will immediately be clear with the idea of what a simply connected domain is; and how is a multiple connected domain defined it is a domain which is not simply connected right.

So, let us look at an example and then you will see how a multiply connected domain basically contains disconnected parts. So, essentially not all closed paths, you cannot keep on shrinking them down to point well always remaining within the domain, right.

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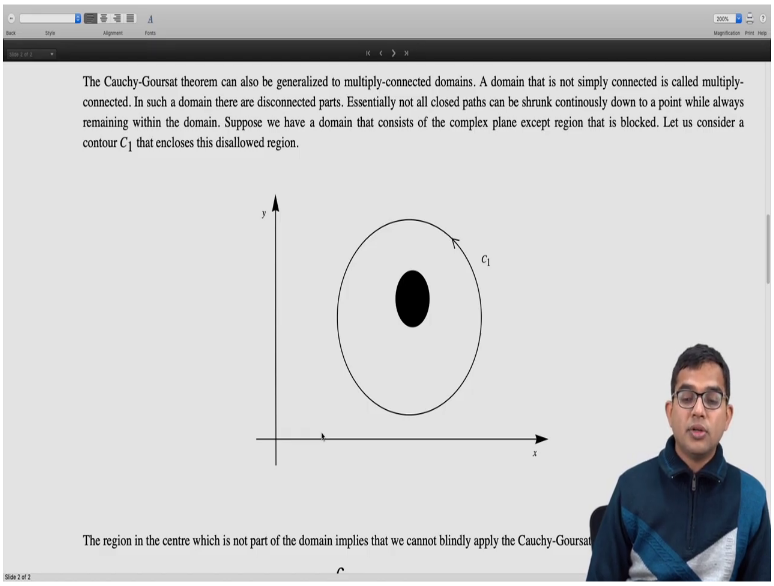

So, suppose we have a situation like this. So, I am looking at a contour C 1 and everything inside this contour is part of the domain except there is some this region. So, in fact, we have an annular region between concentric circles or in this case it does not even have to be concentric circles.

So, there is some sort of mess sitting in the centre. So, there is a non-analyticity here. So, there is a; I mean for whatever reason you are considering the region which is annular to this. You are not allowing you know; you are not allowing this central region to be part of your domain and such a domain is called a multiply connected domain, right.

So, if you consider a multiply connected domain of this kind it can even be just a single point. Suppose, you take the entire complex plane, but do not allow the origin to be a part of your domain, then that is going to be not a simply connected one. Because if you consider circles around the origin and then keep on shrinking these circles in size, you will have to keep on excluding one part of it.

So, you cannot continuously shrink it you know to point to a point and allow it to be part of your domain you know. Everything inside your contour also must be part of your domain; that is not going to happen. So, it is a multiply connected domain.

Now, you cannot apply the Cauchy-Goursat theorem blindly when you have a mess sitting in the centre.

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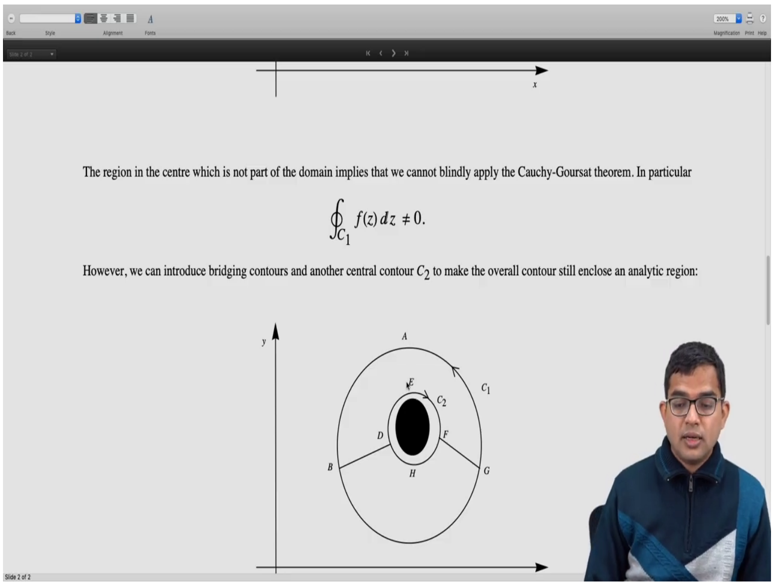

So, in particular you cannot say that this contour integral C 1 f of z dz is equal to 0, right. So, one example we have seen is that of 1 over z squared dz and you cannot blindly say that it is going to be 0 invoking the Cauchy-Goursat theorem, because you know there is a singularity sitting at the origin.

But, we have seen that there is some other way to argue for that particular result and that is why it is important to emphasize it. So, you know that result may hold, but it is not a consequence of the Cauchy-Goursat theorem.

Now, let us see what happens with; how can we still work with a multiply connected domain and still you know that after all this function is analytic in this entire region in this annular region. So, is there a way to come up with some other contour and still make use of the Cauchy-Goursat theorem and what consequences does that have?

So, let us introduce what are called bridging contours. So, you consider some other bit in between contours which I am calling C 2 here and where the direction of motion is now clockwise. So, you can think of this C 1 which is this outer contour and the C 2 as this inner contour.

And so, suppose we consider this entire region which lies between C 1 and C 2. Now, that can be your you know contour of interest. And so, there of course, your function is entirely analytic in this entire region.

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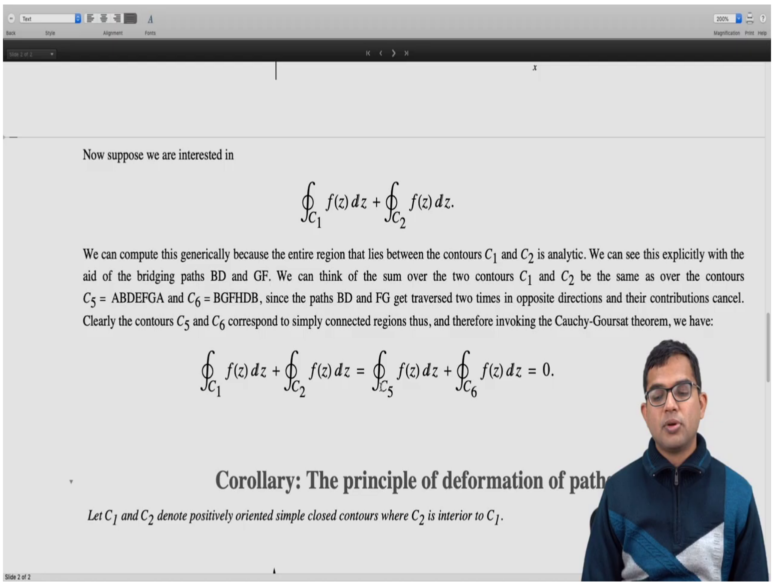

So, you should be able to say that the sum of you know this f of z dz in C 1 plus integral f of z dz C 2 must be 0, right. In some sense you can think of this as a closed contour which you know entirely lies in an analytic region. And so therefore, it should be 0. And, there is a way to argue for this and in a more transparent way and that is to use this bridging contour.

So, what you do is you think of this C 1 plus C 2 is basically you know you go along this. So, you can think of C 1 plus C 2 as the same as C 5 plus C 6. We introduce these contours which we will start with A go down to B ABDEFG and A, right. So, this is one. So, this kind of a contour is one. And then C 6 is something like B BGFHDB, right.

So, you see that if I add this contour and if I add this contour B sorry B BGFBGFDB. So, we see that when I sum these two, I will find that in one of these cases I am moving along this direction from B to D and the other one I am going from D to B. And, likewise in one of them I am going from G to H and in the other one I am going from H F to G, G to F and F to G.

And, these parts will cancel, everywhere else if you just carefully watch what is going on it is nothing but going along C 1 in the outer contour and going along C 2 in the clockwise direction in the inner contour. So basically, and then we argue of course, for this path ABDEFGA.

So indeed the Cauchy-Goursat theorem must hold because everything inside the function is analytic; everywhere inside this region. And it is basically a you know simply connected region and it is a simple contour. So, it is going to be 0 and also in this region it is going to be 0.

So, overall this sum of these two contours is indeed 0, right. So, this is a direct consequence of the Cauchy-Goursat theorem. And, there is an important corollary from this which is also called as the principle of deformation of paths, right.

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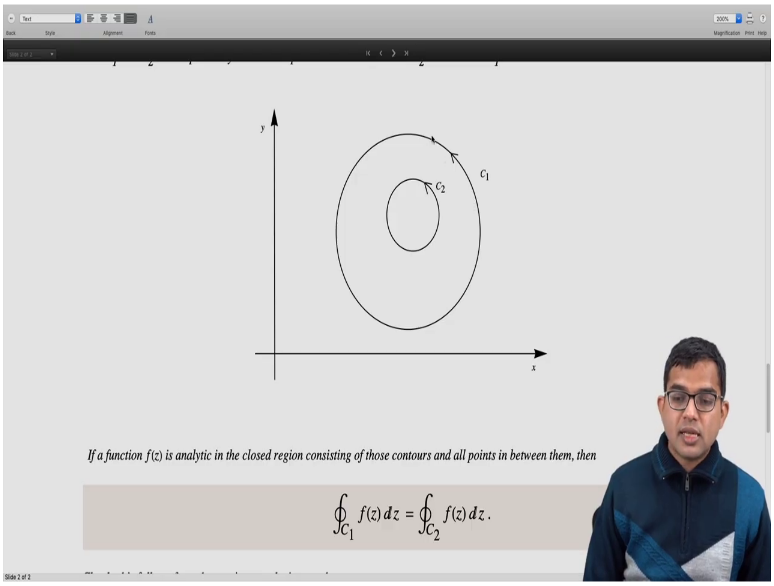

So, if you take C 1 and C 2 to be two positively oriented, right; anti-clockwise oriented simple closed contours; C 1 is some outer one and C 2 is some interior closed contour.

And, then if your function is analytic in this closed region consisting of those contours and all points between them, right. So, we are not saying anything about this inner region. There could be a mess in there. So, what this you know the result pertaining to Cauchy-Goursat theorem for multiple multiply connected domain implies is that; the contour integral over C 1 of f of z dz is the same as the contour integral C 2 of f of z with respect to C 2, right.

So, the reason is that you know whatever mess is there is somewhat hidden right in the centre, right which is interior to C 2. So, then it does not matter which contour you take because you know this in the entire region between C 1 and C 2 it is analytic. On the contour C 1 and on the contour C 2 and in the entire region between C 1 and C 2 your function is analytic.

Therefore, you can actually deform contours, right as long as you do not cross any singularity, as long as you do not cross any non-analyticity. So now of course, you also notice that the sense in which C 1 moves is the same as the sense in which C 2 moves.

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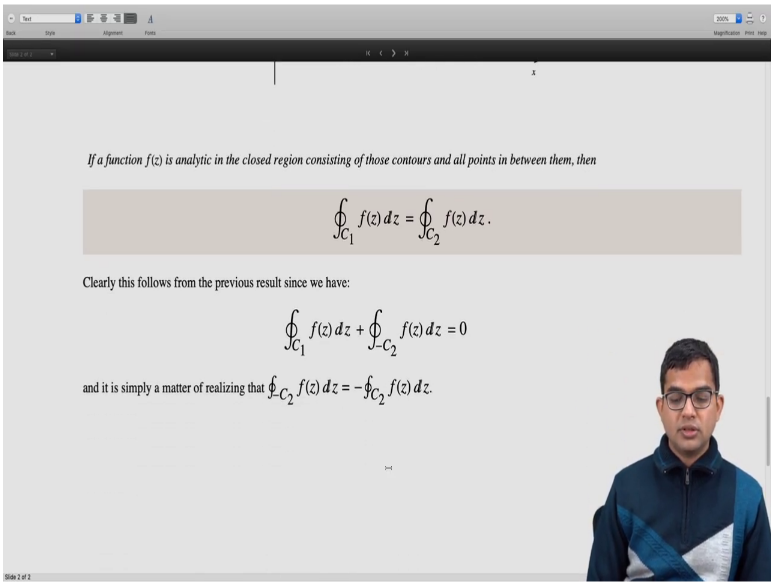

And, this is a direct consequence of the previous result, right. After all the previous result says that you know if you take this and then add it to minus C 2, right; so here you see that the sense goes in the opposite direction when you are doing it as a plus this thing. But then, you immediately argue that plus C 2 you know in this case will be you know minus C 2, because I put it in a different direction.

So, basically what it boils down to is that you can deform contours as long as you do not cross any non-analyticity when you are doing this. So, this is also a result of great importance and we will apply it as we go along as well.

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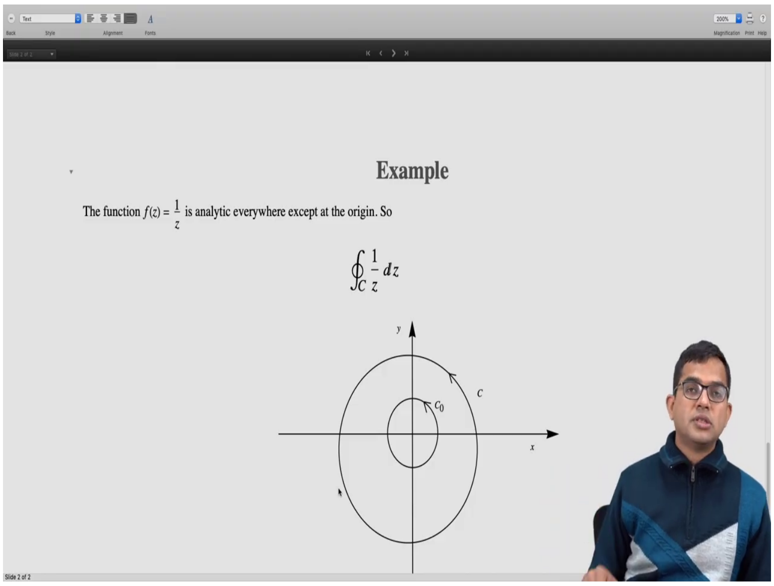

Let us look at an example. So, we have seen that if you consider the function f of z is equal to 1 over z, it is analytic everywhere except at the origin.

And, if we consider this kind of an integral 1 over z dz over some contour C which goes around the origin, right which does not have to be a circle; any contour which is in the positive sense must be the same as you know in the same sense. If you take a different path which we can take for simplicity to be a circle of radius r.

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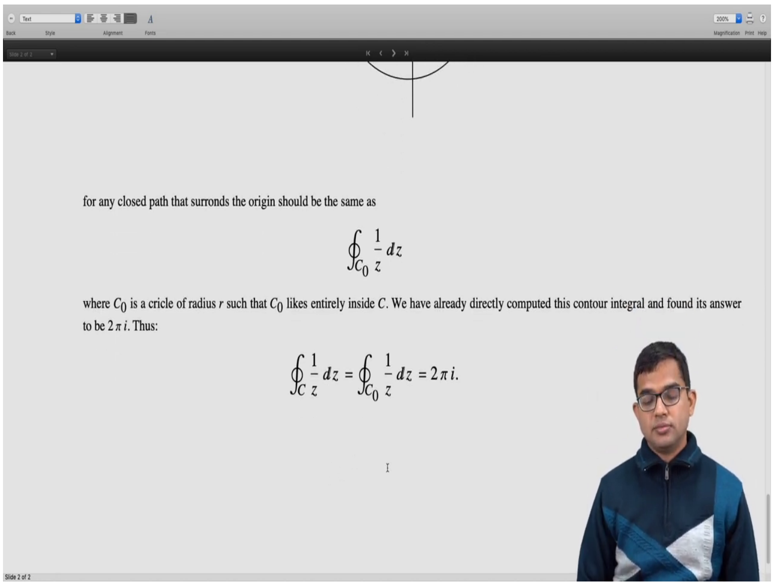

But, we have already evaluated this contour integral of 1 over z dz over C naught and that we have found it to be 2 pi i. So thus, we have the result that 1 over z dz over any closed contour in the positive sense which encloses the origin and does not go through the origin is the same and that is going to be 2 pi i, right, ok. So, that is all for this lecture.

Thank you.