Mathematical Methods 2
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Complex Variables
Lecture - 31
Taylor Series

Ok. So, we have seen how a function that is analytic at some point has derivatives to all orders at that point, right. So, an immediate consequence of this is that a Taylor series of such a function about such a point of analyticity is always available, right. So that is what we will discuss in this lecture.

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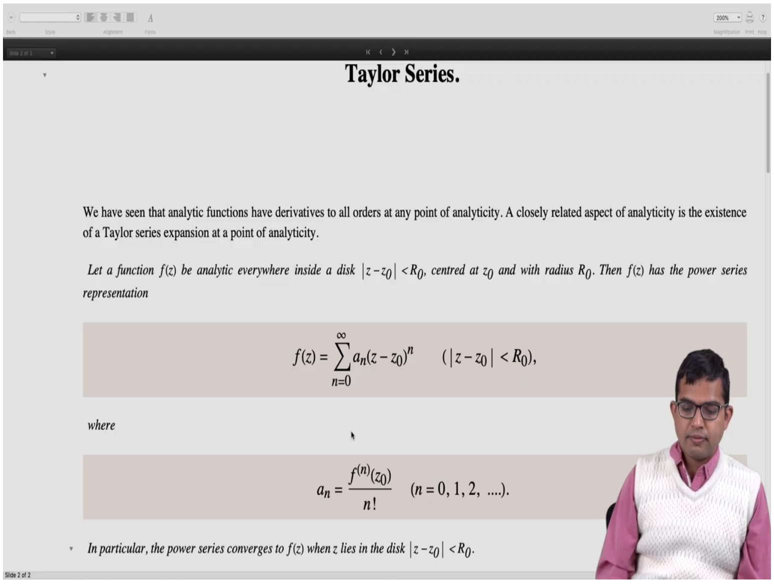

So, if a function is analytic inside some disk and mod of z minus z naught less than R naught which is centred at about this point z naught and with radius R naught, right. So, then f of z is guaranteed to have a convergent power series representation; f of z is equal to summation over n going from 0 to infinity a n z minus z naught the whole power n which is going to be guaranteed to converge inside this disk, right. So, the function is analytic everywhere inside the disk, ok.

So, this function has this power series representation and which is guaranteed to converge within this region mod of z minus z naught less than R naught, and where a n is given by all these derivatives. So, we have already seen that all of these derivatives of all orders exist, therefore it is completely well defined. So, you have an explicit representation for all these coefficients. And there is no difficulty with convergence as long as the mod of z minus z naught lies within this region of analyticity, right.

So, I mean, we will not go into the details of proving this result right, but we will take this as a given, right. So in fact, this is a characteristic of analyticity: Cauchy theorem holds and also the existence of derivatives. So, all orders inside the region of analyticity there is a path independence for these contour integrals, existence of Taylor series, expansions which are convergent, all of these are automatic features of analytic functions, right.

So, even if you are working with a function of a single variable or you know just a function of two variables; let us say to compare with f of z right, so then for a Taylor series expansion to be convergent right. So, you have to work with there are conditions which go into this; I mean you can have functions which are you know once differentiable, twice differentiable or thrice differentiable, but then suddenly the differentiability is lost and so on, right. These kinds of complications are there for functions of a single real variable for instance.

But, on the other hand, if you have a function of a complex variable which is analytic, then you are guaranteed that derivatives to all orders exist and also a Taylor series which is convergent is available.

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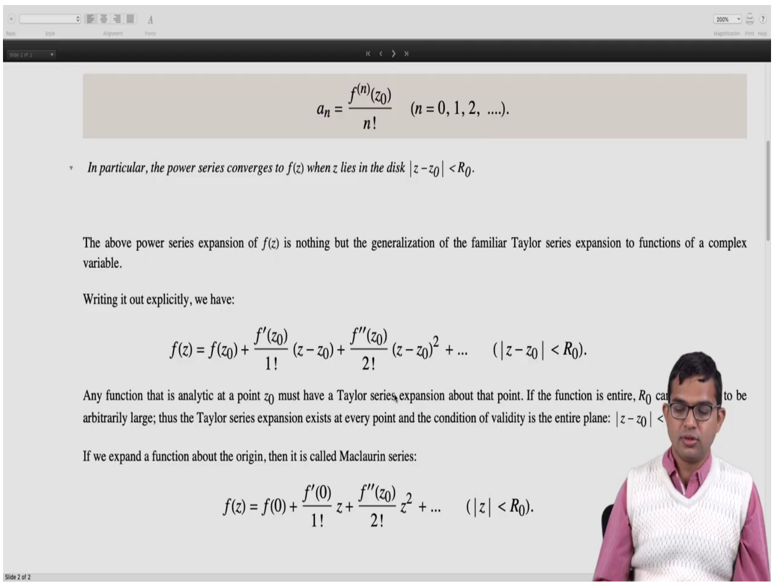

So, of course yeah. So this is really a generalization of the notion of a Taylor series of functions of a real variable, but we see that it is guaranteed to converge, right. And all these inside a region of analyticity, and it is closely related to the fact that you have all these higher order derivatives available.

So, we can write this out explicitly and write it as f of z naught plus f prime of z naught divided by 1 factorial times z minus z naught plus so on right, and mod of z minus z naught is less than R naught.

So, this R naught is something that has to be decided on a case by case basis. So, you have a region of analyticity. So, you are guaranteed that this R naught has to be greater than 0, because you are at an analytic point. We know that analyticity immediately implies analyticity inside a neighborhood around a point; the point of interest. That region of analyticity the radius could be small or big, but it has to be greater than 0 right. So, that is the, that is within the you know very idea of what analyticity is.

And so there are cases where this R naught can go all the way up to infinity, right. So, in particular, if you have an entire function, then you have Taylor expansions available for this function about any point in the plane. And all of this you know different kinds of series that you can write would converge in the entire plane, right.

So, if you expand the function about the origin, then the Taylor series is called a Maclaurin series. And it looks like this f of z is equal to f of 0 plus f prime of 0 over l factorial z plus f double prime divided by 2 factorial z squared and so on, right. Mod z, again you know less than R naught where R naught is this radius of convergence as it is called, ok.

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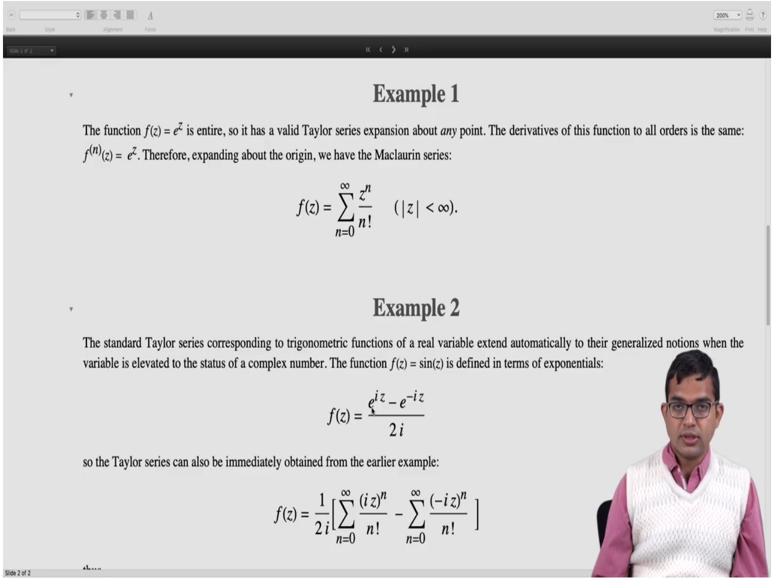

So, let us quickly look at a few examples. So, we have seen that the function f of z is equal to e to the z is entire. So, it has a valid Taylor expansion about any point in the complex plane, any finite z. And the derivatives of this function all orders are also immediately written down. They are simply e to the z. Therefore, expanding about the origin, we have the Maclaurin series as z to the n divided by n factorial, right.

So, you could also work out the expansion about some other point. So, in place of z to the n, you will have z minus z naught the whole power n. And it is just a matter of shifting z to z minus z naught right, and then you will see that this is according to this Taylor expansion. So, let us look at another example.

So, if you want to expand a trigonometric function. So we have seen that trigonometric functions are entire, because after all trigonometric functions sin of z are defined in terms of exponentials both e to the i z and e to the minus i z; that entire it does not matter whether you take the argument to be i z or z, it is complex number exponential of a complex number is well defined for all finite values of z.

And therefore, it is an entire function. So, the linear combination of entire functions is also entire. So, sin of z is entire, cosine of z is entire.

So, sin of z the Taylor series expansion of sin of z can be immediately written in terms of the Taylor series expansions of you know these exponentials, right. So, while in general one has to take special care when you are trying to add you know two series can you add them term by term and all this. But so for our purposes, let us take this to be a given and it is indeed true for this case. And so you can actually combine these two term by term.

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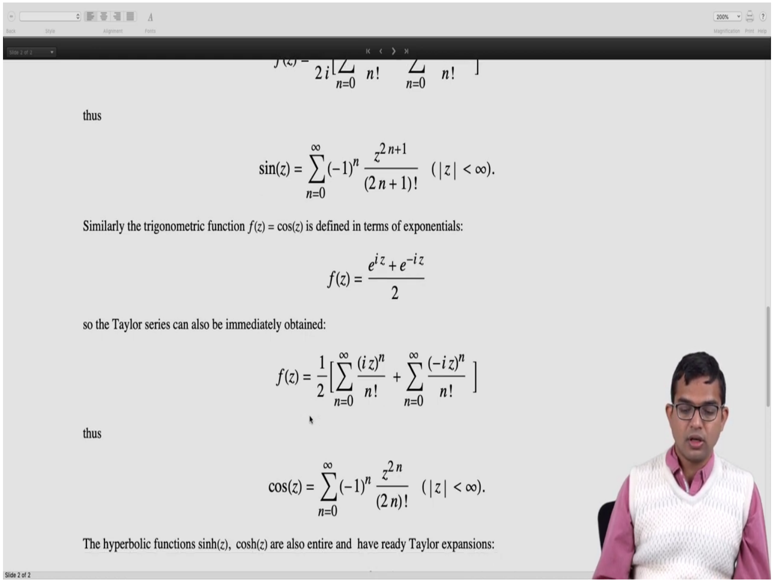

And then you get, i to the i z the whole power n minus minus i z the whole power n and then you get a minus 1 to the n, and then only the odd terms will survive. So, we can as well write this as z to the 2 n plus 1 and a minus 1 to the whole power n comes out. So, it is a matter of you know i to the n minus minus i to the n. So that will become minus 1 to the n, right. So, it will oscillate between the plus sign and minus sign.

So, it is just a matter of you know seeing that whenever you have an even power right, this will vanish i z the whole squared; for example, is just minus z squared, and minus i z the whole squared is also minus z squared. So, it will cancel right. So, but I mean if you take it to an odd power, then you will get a; you get a you know i which will cancel with this i in the denominator, and then you have a minus 1 to the n right. So, it is something that you can quickly verify.

And you notice that sin of z is actually you know just like the expansion we have when you have a you know the sign of a real variable which is very nice. And so, it is just that this z has been now elevated to a complex number. And it is guaranteed to converge for any value of z right. It is very nice, and it is you know it is convenient that our extension of the idea of a sinusoid you know works out so nicely also for the Taylor expansion.

And likewise cosine of z you can check directly from first principles from definition of what cosine of z is you know as a sum of these exponentials, again some of these two different series. And then you get the cosine of z is equal to summation over n minus 1 to the n z to the 2 n divided by 2 n factorial.

Once again it is an entire function, so mod z less than infinity it is going to be convergent. You can also shift this point about which the expansion is considered for to any other finite z naught in the complex plane.

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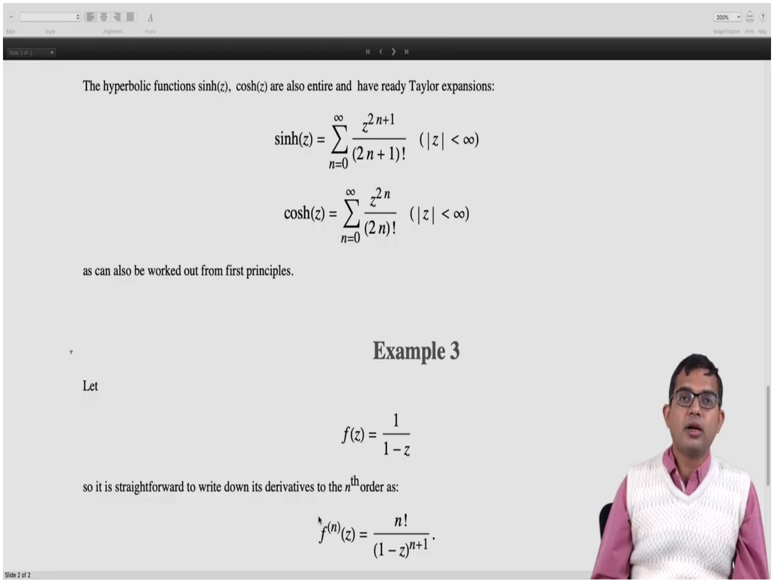

So, once again cipher hyperbolic functions, sine hyperbolic of z, cosine hyperbolic of z are also you know readily expanded in terms of Taylor expansions. Both of these linear combinations of exponentials are also entire functions. And they have these Taylor expansions which are completely you know natural generalizations of their corresponding Taylor expansions for you know corresponding functions as a real variable, ok.

So, let us look at another example. Now, this time we look at an example of a function which is analytic, but it is not entire - it is analytic in a region right. So, we see that one over 1 minus z is you know it has a it is analytic everywhere except at the point z equal to 1.

So, if you are looking at you know its derivatives, it is straightforward to write down these derivatives explicitly, the n-th order derivative is going to be just n factorial divided by 1 minus z the whole power n plus 1. And again this is well-defined at all points other than z equal to 1.

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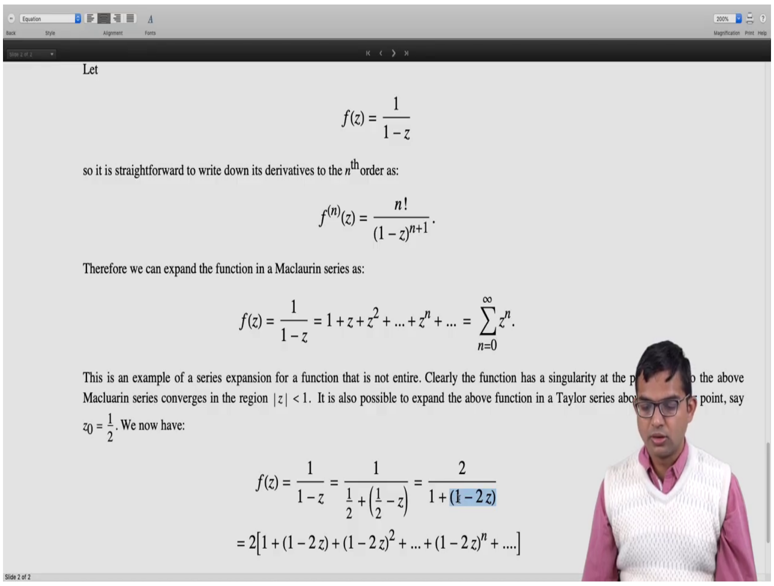

And so we can write down this Maclaurin series f of z is equal to 1 over 1 minus z is just 1 plus z plus z squared all the way up to infinity, right. So, summation over n, n going from 0 to infinity z to the n is this function, right. And this function is not entire, and it has a singularity sitting at z equal to 1. So, for all mod z less than 1, this particular series expansion will converge, and it will converge exactly to this function, right. So that is the you know statement of the Taylor-Maclaurin theorem if you wish.

And, it is also possible to expand the same function about some other point inside this region of analyticity. For example, you could take this z naught to be a half. And we now have it is convenient actually to rewrite it in this form and use the same result that we already had.

Also you can directly work it out from first principles you know work out all these derivatives evaluated at the point z, z naught equal to half and then you can actually cross check that you get back the same result. So that is another way of doing it; f of z is 1 over 1 minus z which is it can be written as 1 over half plus half minus z, and then you pull out this factor of 2 from the denominator; that factor of half from the denominator which becomes a factor of 2 in the numerator, and then you have 2 divided by 1 plus 1 minus 2 z.

Now, you can actually think of this 1 minus 2 z as a complex number, right. So, instead of thinking about f of z, it can be f of w where it is 2 divided by 1 plus w, right. And then you can expand this function or you know in terms of 1 1 over 1 plus w right, so in place of minus z you have a plus w.

So, you get 1 plus 1 minus 2 z, so you have to be a bit careful. So it is 1 minus; so it is actually 2 z minus 1 plus 2 z minus 1 double squared and so on. So, let us check this. So, 1 minus z is good. So, it is 1 plus 1 minus 2 z. So, if you expand it, now this is more like 1 minus 2 2 z minus 1.

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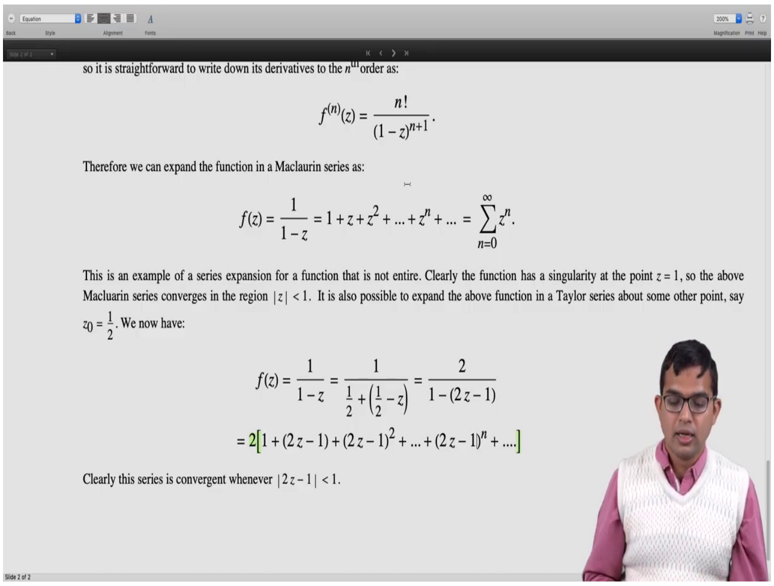

So, this should be, it is better it is more convenient to rewrite this as 2 z minus 1, and then you have with an overall minus sign, with an overall minus sign, so it is a minus 1 minus 2 z minus 1 and then so that you if we can immediately connect to this z here.

And so now, we have 2 z minus 1, should be 2 z minus 1, 2 z minus 1 everywhere, 2 z minus minus 1 the whole squared so on the whole power n so on right. And clearly this series is convergent whenever mod of 2 z minus 1 is less than 1, right. So in fact, you can try to expand the same function about some other point.

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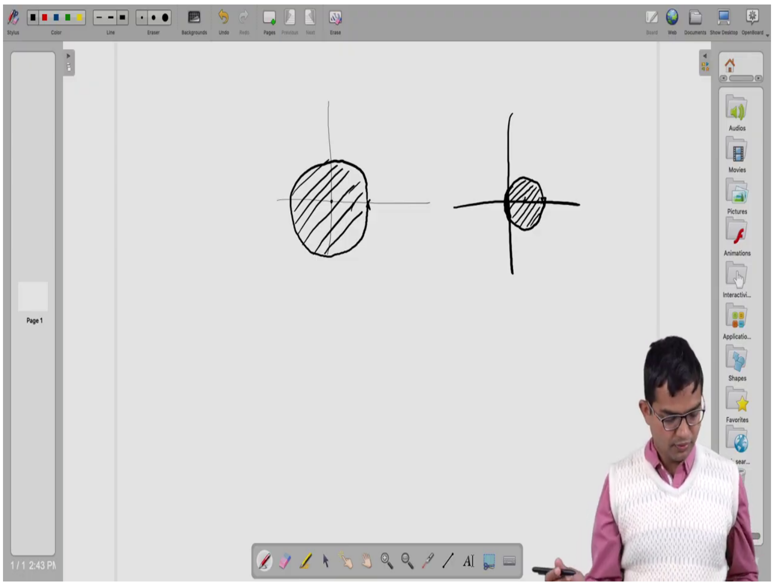

There is a singularity sitting at the point z equal to 1. So, if I expand about z equal to 0, then I have found that this series is convergent in this entire circle of radius 1 in mod z less than 1, so in the interior of the circle.

But on the other hand, it for the same function if I were to expand about this point; point half. So, then I find that it is convergent in a smaller circle of radius; it is supposed to come here of radius half.

And so the same function has a different Taylor series if you are expanding about a different point, and it has a radius of convergence which is half, right. So that is what we have found here z 2 z minus 1 mod of this must be less than 1, right.

So, in this lecture, we have looked at how the idea for Taylor series can be generalized to functions of a complex variable, we have looked at a few examples of analytic functions some of which were entire, and some other examples where the function was analytic inside a region.

And how the Taylor series is always available for an analytic function about a point of analyticity and whose convergence is guaranteed inside that is inside the neighborhood of an analytic point as long as there is no singularity there. So, inside this whole region of analyticity around a point of analyticity, this Taylor series is guaranteed to converge, ok.

Thank you.