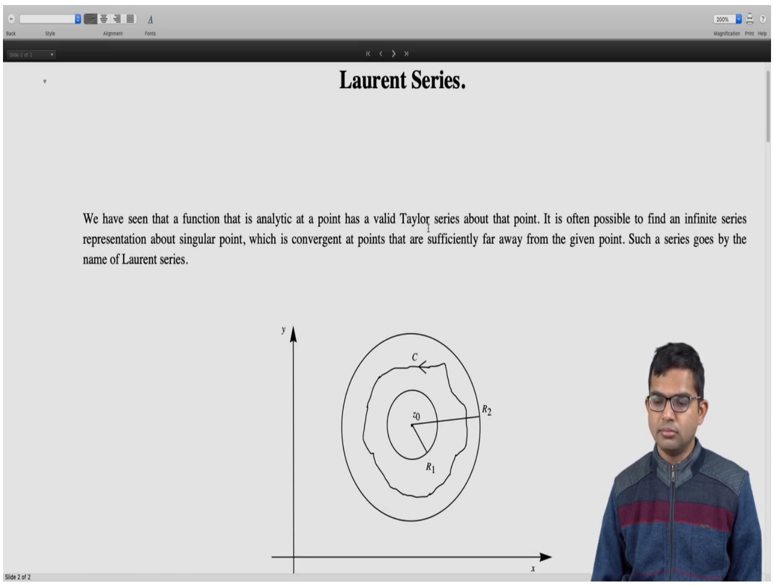
Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Complex Variables
Lecture - 32
Laurent Series

Alright. So, we have seen how a function which is analytic at a point has a well-defined Taylor series and you know which is convergent in some region around that point of analyticity. So, in this lecture, we will see that in fact, it becomes possible to write down a series even about a non-analytic point.

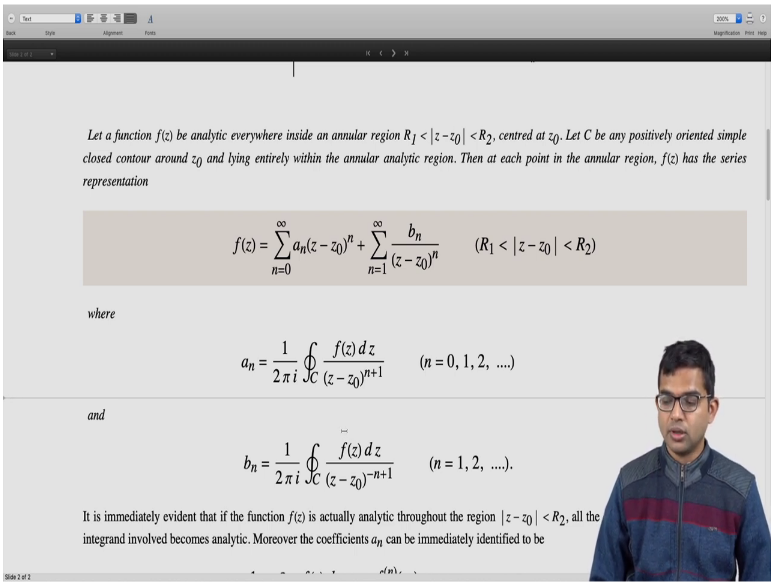
So, we have seen that you know regions of analyticity are common, but there are also points of non-analyticity or regions of non-analyticity which are very common with lots of functions of complex variables and its useful often to be able to write down a series which is expanded about a point of non-analyticity, but which is valid in some region of analyticity which is far away typically from this non-analytic point.

(Refer Slide Time: 01:11)



So, the details of which we will discuss in this lecture ok. So, suppose we look at these two circles that I have drawn. So, there is a circle of radius R 1 which is centered about this point z naught and then, there is another circle of radius R 2 which is also centered about this point z naught.

(Refer Slide Time: 01:37)



So, we are given that there is a function which is analytic in this annular region. So, as long as mod z minus z naught lies between R 1 and R 2, this function is analytic right. So, when this is the case and so, you can consider some arbitrary contour right, which can have you know pretty much any shape and which is a simple closed contour of this kind.

And which is positively oriented and which lies entirely within this region of analyticity. Then, at each point in the annular region, so f of z has a series representation right, so that we can expand you know summation over n going from 0 to infinity a n z minus z naught the whole power n which looks like the Taylor series.

But then, you will also have these kinds of terms summation over n going from 0 to infinity b n over z minus z naught the whole power n right. So, where, a n is so actually this summation the second summation will turn from 1 to infinity right. So, b n would be defined all I mean there is a constant term which is already included in the first series, the second series will run from 1 to infinity alright.

So, all powers of z minus z naught which lie in the denominator 1 over z minus z naught. Now, the key point is that these coefficients a n can be written down in terms of these contour integrals. So, it does not matter which contour you take right.

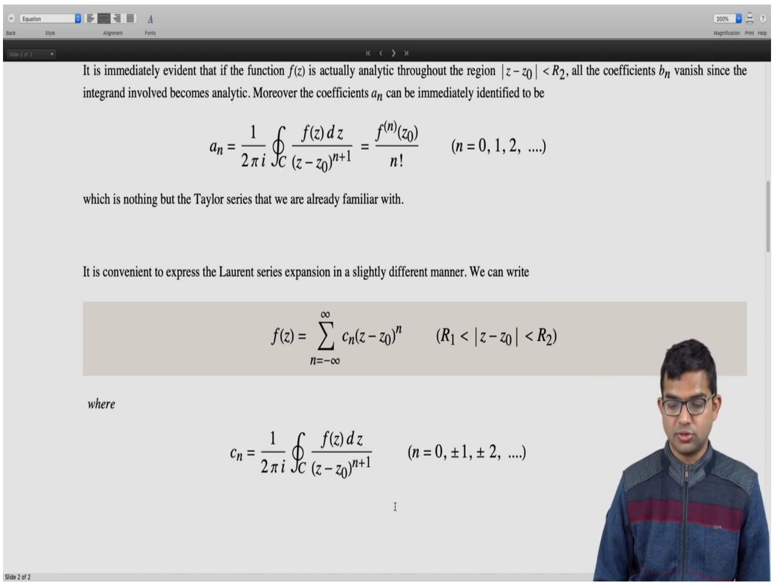
So, this is what we learned when we studied the Cauchy's integral formula right. So, whether you take a contour of this kind or another contour it does not matter. So, there is this principle of deformability of contours. So, these coefficients are written in terms of these contour integrals where you know you divide f of z divided by a suitable power and then, you integrate.

And so, basically what happens is all the other coefficients you know, all the terms corresponding to other coefficients will vanish, when you are dividing by you know exactly the power which is useful for your case when some ways you are isolating the term you know corresponding to a n and then, you know this relation holds.

And so, what is remarkable is, so I mean this is you know exactly like the Taylor series itself is nothing but the Taylor series; whereas, you know these are the you know terms which are new in some sense. All these b n represent the singular region.

So, there is this inside region which has an 1 analyticity or may or may be more than 1 analyticity non-analyticity and so, that is you know that is the point about which you are trying to find this series expansion and such a series expansion, it goes by the name of Laurent series.

(Refer Slide Time: 04:47)



And so, it is in fact, convenient to first of all acknowledge that these a n's are actually nothing but the coefficients which would come out from a Taylor series expansion. That is why we also recognize that there is this nth derivative of f of n of z naught divided by n factorial which is what would happen if R 1 is actually 0 right.

So, it can go down to if you know this annular region in the limit can you know go can be squished down to 0 which means well I mean the inner radius can be squished down to 0's. The annular region will actually become the entire circle which is or a circular region which is an enclosed by the circle of the larger radius right.

So, when that happens, then we recognize that these a n's are actually nothing but the nth derivative at the point z naught divided by n factorial. But in general, I mean if the function itself is not well-defined at that point at z naught, so there is no question of its derivative having a meaning.

But if the whole region is a region of analyticity, then you know this term is recognized to be actually nothing but the nth derivative of the function at that point right. So, then, you have only the Taylor series and all these coefficients b n will vanish right, when that happens right.

So, it is convenient to actually think of this Laurent series expansion as just you know this kind of a series; summation going from n equal to minus infinity to plus infinity c n times z minus z naught to the whole power n. So, now, n can also take negative values.

So, it is basically the same thing and then, there is just one common expression c n is equal to 1 over 2 pi i. You know this contour integral c; whereas, where c entirely runs in the analytic annular region, then you have f of z divided by z minus z naught the whole power n plus 1 d z and now, n can take all values all integer values 0, plus or minus 1, plus or minus 2, so on right.

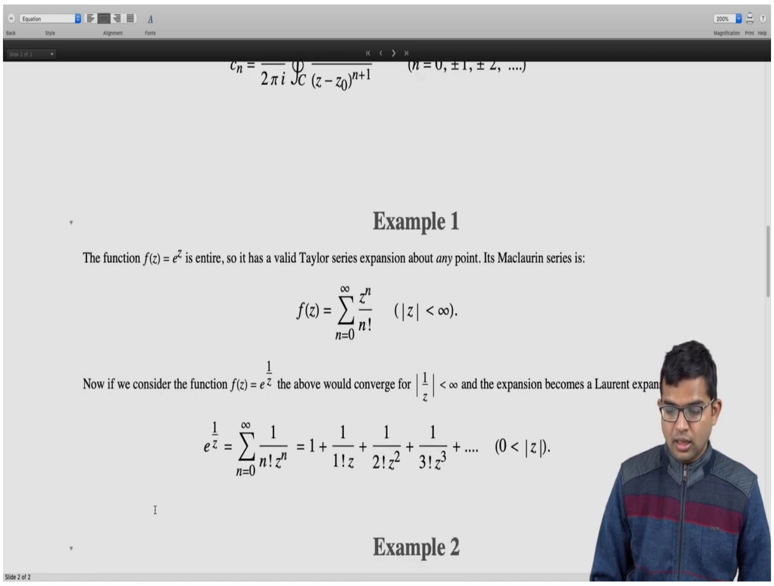
So, the key points to emphasize here are you know these a n's and b n's are you know valid even when you have a genuine non-analyticity sitting at the you know at the about the point which the expansion is happening; but also when the entire region is the region of analyticity right.

So, then, you have this becomes the Taylor series expansion and all these coefficients b n simply vanish when that happens right. So, one way to see this is. So, if f of z itself is an analytic function, then if you. So, this dividing by z minus z naught to the whole power minus n plus 1 is basically like multiplying by z minus z naught to n minus 1.

So, you are multiplying an analytic function by another analytic function and then, Cauchy's theorem tells us that this contour integral has to be 0 right. So, it is actually quite straightforward to see that if the function is analytic everywhere inside you know the region bounded by this simple closed contour, then for sure all these coefficients b n s simply vanish.

So, that comes as a direct consequence of the Cauchy theorem. But what is remarkable is that we have this very beautiful expansion available even about a non-analytic point right. So, the only thing to take care of is to identify these radii R 1 and R 2 ok. Let us look at a few examples.

(Refer Slide Time: 08:51)



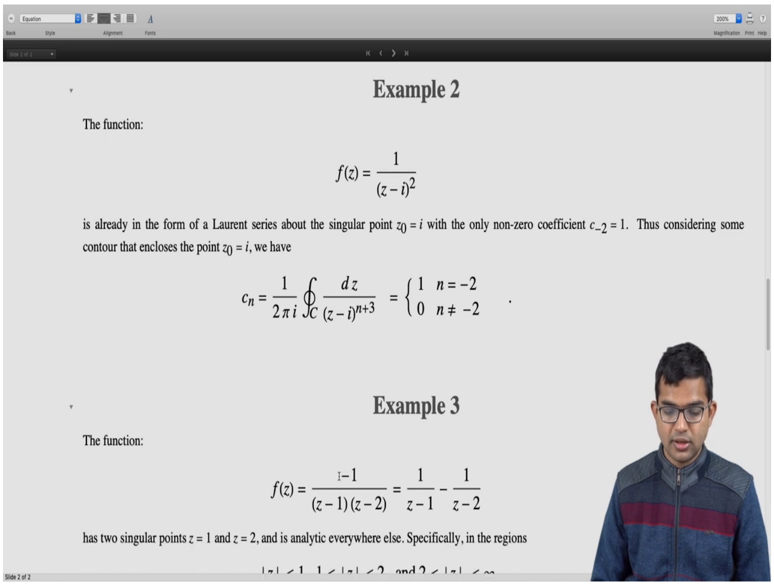
So, perhaps, the simplest example is the expansion of the function e to the z. We know that e to the z is entire and it has a valid Taylor series expansion about any point. So, its McLaurin series in particular, we have already seen: f of z is equal to summation over z to the n by n factorial; n goes all the way from 0 to infinity and this series expansion is valid at any point z which is in the finite complex plane.

So, now, if we consider the function e to the 1 over z, now the above you can think of you know 1 over z as some w which is a complex number and this would also have an expansion about you know w equal to 0, where mod w must be less than infinity. According to this, exactly this and then, the expansion actually is what is a Taylor expansion in w is really a Laurent expansion in z right.

So, you can write it out in one step and then, you have this whole expansion and we see that every point other than z equal to 0 must be yeah you know, so this is going to be a convergent series right. So, what was infinity in the earlier example is basically the same role played by 0.

So, there is this you know what is called an essential singularity sitting at the origin right and so, you have you know terms to all orders in 1 over z and so, for any mod z greater than 0, this is going to be a convergent series. So, this is a very simple example of a Laurent series in which the annular region, if you wish, is really it runs all the way from 0 to infinity; any point greater than 0, all the way to infinity is the annular region right. Although, it is not perhaps the most natural way to think of this mod z greater than 0 as an annular region.

(Refer Slide Time: 10:47)



Let us look at another example. So, here, it is actually already explicitly already in the Laurent expansion form because I mean I have this function 1 over z minus i the whole square. So, there are no other terms. It is the Laurent series about the point z naught equal to i and there is only 1 coefficient which is nonzero and that is c minus 2; every other coefficient is 0.

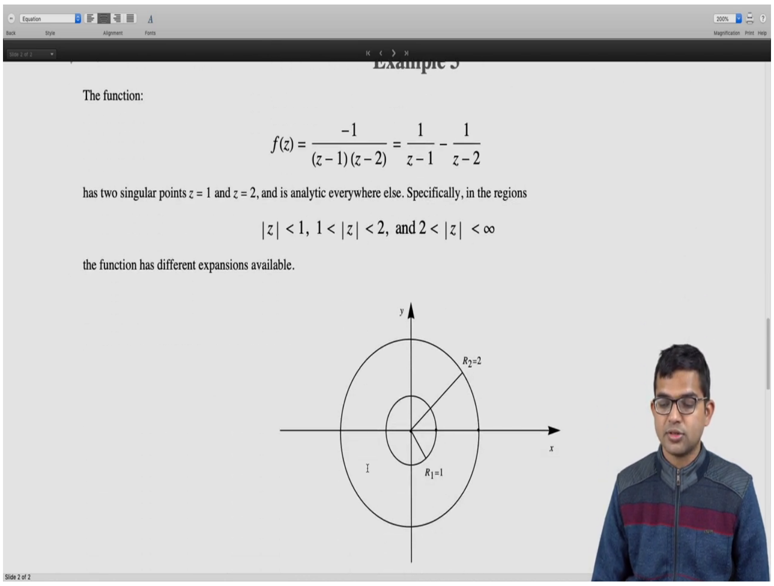
And so, we have I mean we can verify that c n is 1 over 2 pi i integral contour integral about this point d z divided by z minus i the whole to the n plus 3 and so, whenever n is equal to minus 2, then you get a 1, everywhere else you get 0 right. So, this is something that is a result we have already seen right.

So, whenever you take a point like d z and divided by z minus i, z minus i and do a contour integral that is going to give you 2 pi i. These are 2 pi i sitting outside. So, it will give us equal to 1 and every other power of z minus i is going to and in this form of an integral is going to give you 0. This is a result which we have already seen right. So, this is a straightforward example of a Laurent series about in some sense a trivial example of a Laurent series which involves just one term.

So, let us look at another example which illustrates how you can have you know an inner radius and an outer radius and it comes out in a very nice way. There is another example, where we have I guess two explicit singularities; one of them is sitting at z equal to 1 and the other one is at z equal to 2. I mean you can start with this function 1 minus 1 over z minus 1 times z minus 2, but you can easily expand it to this partial fraction expansion as 1 over z minus 1 minus 1 over z minus 2.

So, you see that this is not really a Laurent expansion, it is just you have two different terms. If it had only one of these, then it would be the Laurent series expansion about that particular point. But you have 2 of these, so then, we have to be you know you have to first of all isolate the region of interest. So, there is a singular point sitting at z equal to 1.

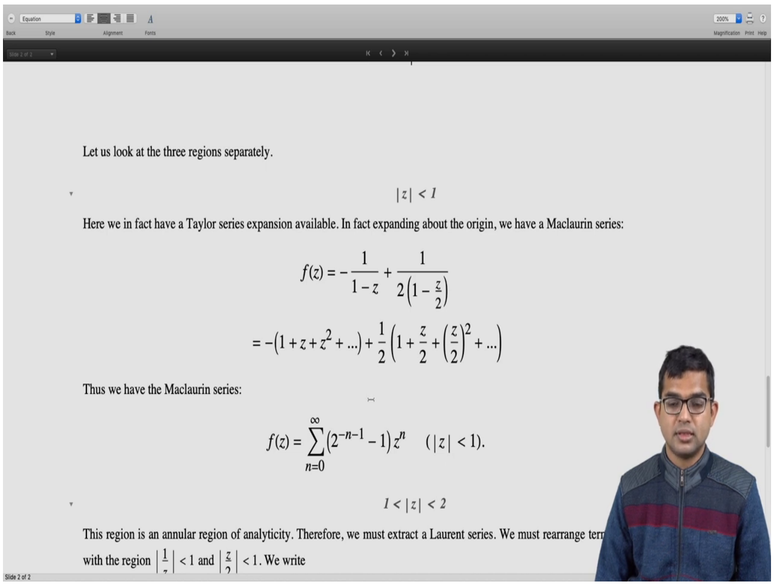
(Refer Slide Time: 13:01)



And there is another singular point sitting at z equal to 2 and everywhere else, this function is analytic. So, in fact, automatically we have these three regions of interest mod z less than 1, mod z lying between 1 and 2 and mod z greater than 2 and less than infinity right. So, I will repeat the picture here.

So, mod z less than 1 you know this is in fact going to be a Taylor series. So, let us try and expand this function in each of these three analytic regions. So, for mod z less than 1, you actually get a Taylor series or a McLaurin series, if you are going to expand it about and the origin, we are going to expand it about the origin.

(Refer Slide Time: 13:43)



So, we can simply write this down as minus 1 over 1 minus z and then, since we want to consider mod z less than 1, it is convenient to write down the second term as plus 1 over 2. Pull out the 2 and you have 1 minus z by 2. So, we have mod z less than 1. So, clearly, mod z by 2 is also less than 1 right.

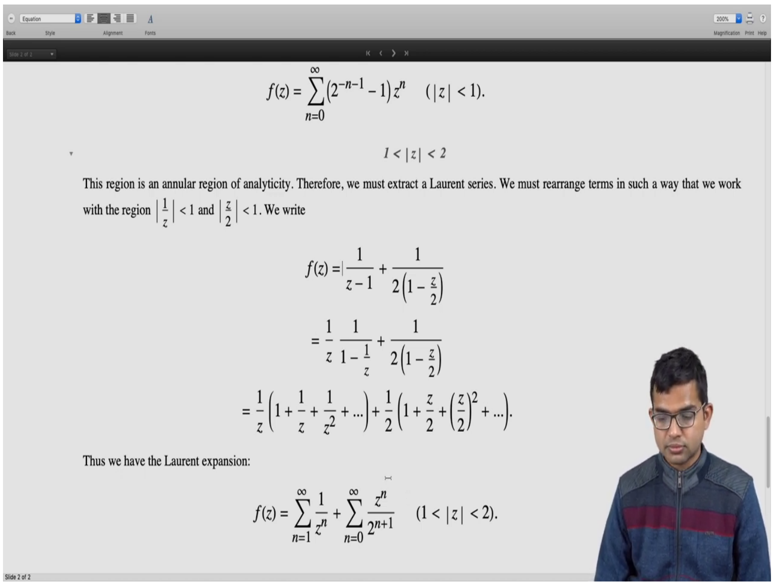
So, mod z is less than 2. Therefore, each of these have this familiar series expansion. So, it becomes 1 minus or 1 plus z plus z squared plus so on and then, we have a plus half times 1 plus z by 2 plus z by 2 to the whole square plus so on, that is an infinite series.

This is an infinite series and it turns out that you can add these kinds of infinite series term by term and then, you get 2 to the minus n minus 1 minus 1 the whole times z to the n and z goes from 0 all the way up to infinity and this is a convergent Taylor or a McLaurin series in the region mod z less than 1.

Now, the region between where mod z is between 1 and 2 is an annular region. So, this is you know exactly like the picture, we first flashed when we discussed how a Laurent series is possible. So, here is the region of analyticity and we are expanding about the origin right. So, which is.

So, although the origin itself is actually analytic here, the point is that you know we are looking at this region between 1 and 2 as your region of analyticity and you are expanding about the point z is equal to 0. So, that is why this classifies as a Laurent series as well and then, there is a non analyticity at z equal to 1 right.

(Refer Slide Time: 15:41)



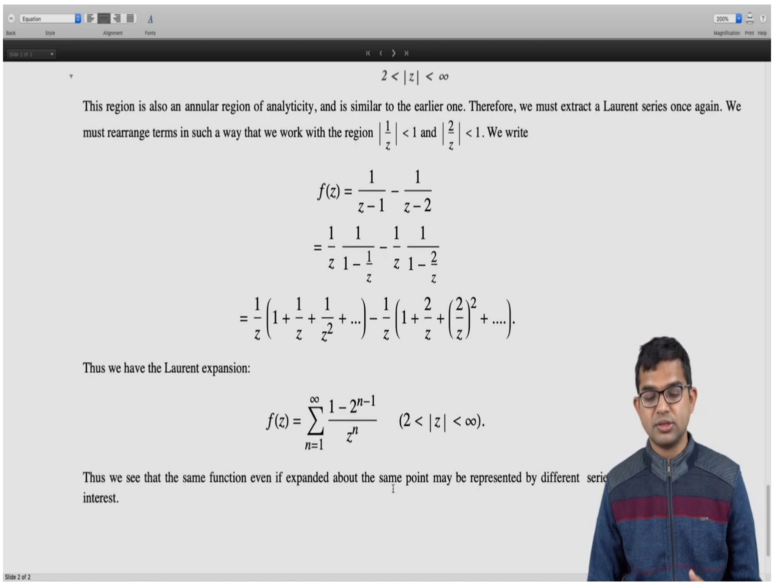
So, let us see how this plays out. So, the trick here is to rearrange it in such a way that you are true to the region of interest right. So, if you are in the annular region. So, mod z is greater than 1. So, it is useful to think of this as mod of 1 over z is less than 1 and again, mod z by 2 is less than 1 right.

So, mod z is less than 2. Therefore, mod z by 2 is less than 1. So, it is convenient to write down this function in this form, you know you leave 1 over z minus 1 and then, pull out the z and then, you get a 1 over z times 1 1 minus 1 by z and then, you have the second part really is left unchanged.

Because after all you are interested in the region mod z by 2 is less than 1 and then, we use the same rule since 1 over you know mod of 1 over z is less than 1. So, you have this expansion 1 over z times 1 plus 1 1 over z plus 1 over z squared so on expansion and then, the second one is basically the same because you are still in the region mod z by 2 is less than 1. So, then you have to stitch these 2 together and you get this Laurent expansion now.

So, this is like a genuine Laurent expansion which involves powers of 1 over z and then, you also have powers of z. So, 1 over z to the n summation 1 to infinity and also, you have a summation n equal to z to the n over 2 to the n plus 1. So, this expansion is going to be valid only in the region between 1 and 2.

(Refer Slide Time: 17:17)



And then, you have the third region of analyticity which also can be thought of as another annular region. So, as far as the region 2 less than mod z less than infinity is concerned, you have these conditions mod of 1 over z is less than 1 and mod of 2 over z is also less than 1.

So, now, it is convenient to actually pull out 1 over z from each of these two terms in the function. So, we write it as 1 over z times 1 minus 1 over 1 minus 1 over z and minus 1 over z times 1 by 1 minus 2 over z and then, you simply expand using the one you know familiar series expansion which is like a geometric progression.

So, 1 over z times 1 plus 1 over z plus 1 over z squared so on all the way up to infinity minus 1 over z times 1 plus 2 over z plus 2 over z the whole squared plus so on. So, thus, we have a different Laurent series expansion now. It is all in power. So, 1 over z and so, this is a Laurent series expansion which is valid in this other annular region, where 2 is less than mod z less than infinity.

So, what we see from this set of examples is that even though you are really expanding, if you are finding series expansion of the same function about the same point, but in different regions. So, you get different series and different kinds of series depending upon the region in which you know you expect these series to converge right. So, in this lecture, we have looked at the idea of a Laurent series and how you know the region in which the series is valid must be picked carefully ok.

Thank you.