Mathematical Methods 2

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Complex Variables
Lecture - 33
Convergence

So, we have been looking at a bunch of series, right. So, we looked at Taylor series then we looked at the Laurent series which is typically defined about a point of non analyticity or you know a point which is centred in a non analytic region and then it has its own region in which this series would be convergent. So, let us look at the idea of convergence a little more closely in this lecture, ok.

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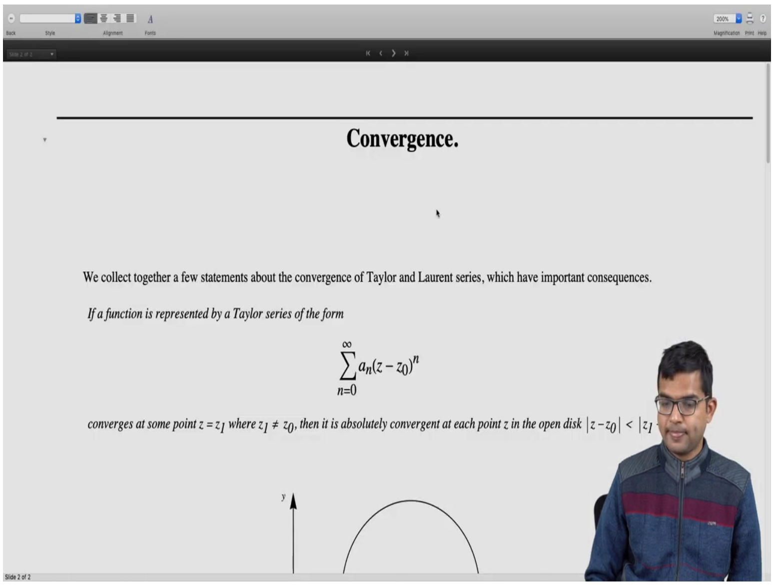

So, we collect together a few statements not necessarily trying to provide any rigorous proof or anything, but these are all you know well established arguments. So, our goal will be to assimilate the various different facts associated with Taylor series and convergence associated with them and we wish to be aware of these. So, that we know you know where these things hold and when they may turn into difficulties, right. So, that is the philosophy.

So, if a function is represented by a Taylor series of this form. So, suppose we just know the Taylor series of some function and then we know we happen to know that this converges at some point z 1 other than z equal to z naught. Of course, if there is a Taylor series of this form and if it's a valid Taylor series then it must definitely converge at z naught.

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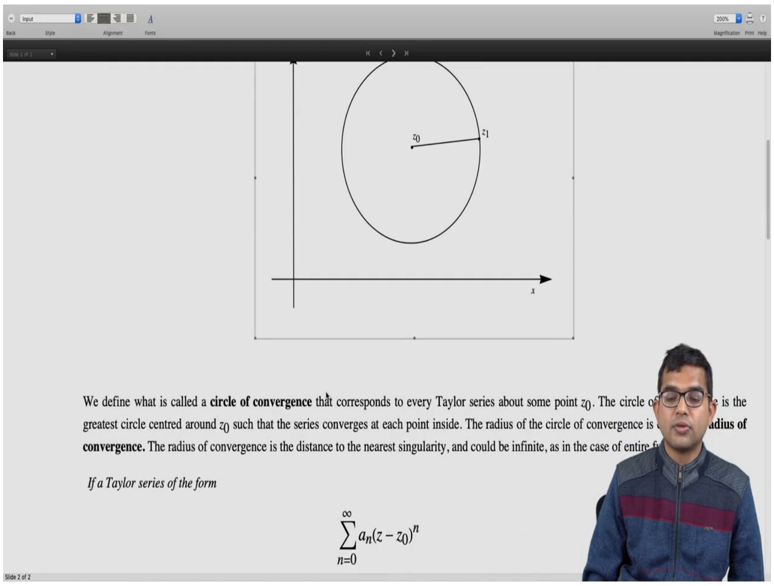

And if we happen to know that there is some other point z naught at which it also converges then it is possible to show that. In fact, this series is absolutely convergent, not just convergent, but absolutely convergent.

Absolutely convergent simply means that in place of you know these; so, you can think of this as a bunch of complex numbers which are being added, but in place of these different complex numbers which are being added you simply replace each of these you know terms in your yeah in your series expansion by that absolute value, right.

Even that is going to converge at every point that lies in the interior of this region which is defined by the circle centred about z naught and whose radius is mod of z 1 minus z naught, right. So, that is a theorem which says that and it is not very difficult to prove. I mean you can show that there is a certain series involving real numbers really.

So, when you take modulus of this then you can argue that this is a convergent series the absolute you know the sum of the absolute values, whenever z lies between you know its it lies within the circle defined by z 1, right. So, we will not go into the details of this argument, but basically it's true, right. So, what it boils down to is if you are given a Taylor series about some point and if there is another point z 1 which is some distance away from it.

Then for sure you know this in this entire region which is interior to this you are guaranteed that this Taylor series will converge and closely associated with this is the concept of the circle of convergence, right. So, this is defined as the greatest circle around z naught such that the series converges at each point inside. So, given that you have a Taylor series which is well defined at a point for sure it is well defined in some circle of convergence.

So, maybe a small circle or a big circle and it may be even a circle with radius infinity. So, the radius of the circle of convergence is called radius of convergence right and it could be infinite; that is the case when you have entire functions.

We have seen entire functions. A classic example is of course, e to the z now e to the z has this expansion which is valid about any point. Let us say you are looking at it about the origin then this expansion will converge no matter what value of z you take, any finite value of z and it is going to converge, right.

But, on the other hand if you look at other kinds of functions, we will look at some examples, where there is going to be a finite radius of convergence, finite you know circle of convergence and if that happens then I mean typically there is going to be a singularity sitting on this circle, right.

It could be one singular, it could have been more singularities right, but there is at least a singularity sitting there. And what this tells you is that you know the definition of the circle of convergence along with you know this idea that if you can find any point z 1, where it is convergent then you know all the entire region interior to that is also convergent.

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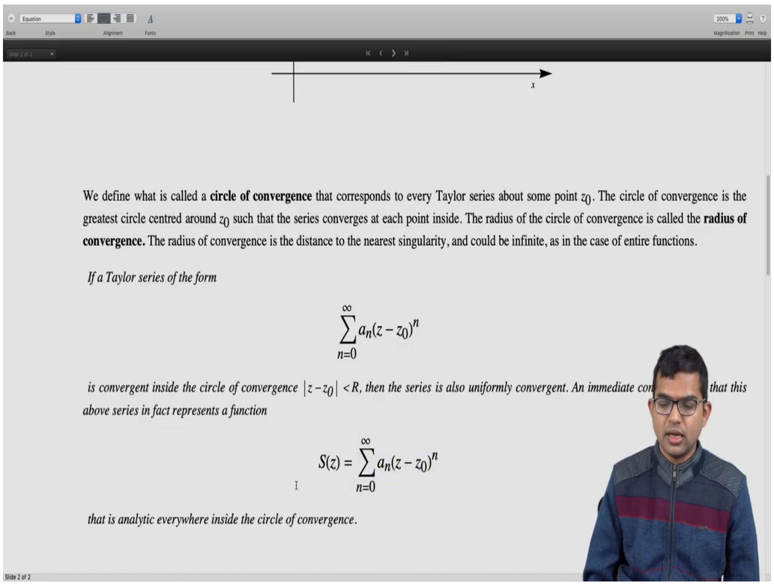

Means that if there is a circle of if you found the circle of convergence then for sure any point outside the circle of convergence which will not be a convergent point for this series.

Because if it is then by this theorem then it would mean that you know that will become the radius of convergence not something less than that right because of this fact which we just state. So, there is a circle of convergence and any point outside the circle of convergence cannot be a convergent point, right.

So, if I; so, then there is another idea which is of great importance and that is the idea of uniform convergence, right. So, again we will not be giving any rigorous proofs or anything of this kind. So, the idea of uniform convergence is basically that you know if you take a whole region in which your Taylor series is convergent you have a series of this kind and it is convergent in this entire region, mod of z minus z naught less than 1.

Now, if you take any neighboring points, basically the series will converge in pretty much the same manner at any two neighboring points, which is a very crude way of saying it. But, basically what essentially it means is you can think of you know this function S of z, it is a continuous function.

You know the idea of convergence of a series is that you know subsequent terms should become lower and lower and then they should 0 in on a certain value. So, you should be able to find given any if its if you are saying that this is this series at a point z converges to a value

S of z then you should be able to get as close as you want to this point by choosing a you know a term truncation of this term you know which is sufficiently large.

So, what the idea of uniform convergence means is that given that you want you know S of z to be you know close to the you know you have given a certain spread around this S of z at a certain point and if you give the same spread at a neighboring point right its enough to truncate the series at the same n in both of these cases basically.

I mean that is a crude way I am putting it. I am not going to give you the details, but essentially what it means is that this kind of a series is going to give you a continuous function. S of z is a continuous function and in fact, it's more than continuous.

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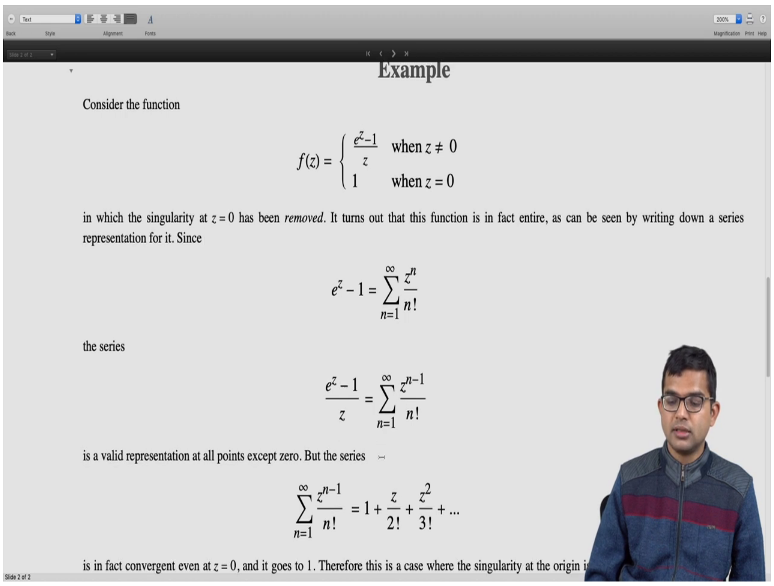

It's actually analytic everywhere inside the circle of convergence and that has important consequences because then you can take the derivative of such a function you can work with contour integral sort of term by term and all this. We will discuss this as we go along.

But for now you know we are stating these as facts and let us look at an example to illustrate how this works out. So, suppose you consider a function like e to the z minus 1 over z, right. So, this function clearly has some mess sitting at z equal to 0 and. So, at exactly z equal to 0 you define this function to be 1 and this function to be e to the z minus 1 over z everywhere else.

So, now it turns out that this function which appears to have a singularity at z equal to 0 has actually been cured of the singularity. You say that there is a removable singularity and that singularity has been removed, right. So, it turns out that in fact, this function defined in this manner is actually an entire function as can be seen by writing down this series representation and then just using the arguments which we just presented earlier, right.

So, first thing to do is to just expand e to the z itself which we know how to do and then e to the z minus 1 will turn out to be just summation over n going from 1 to infinity z to n over n factorial. And so, then if you divide this whole thing by z then you get another series where you know the series starts from 1, right. So, e to the z minus 1 1 over z this is 1 plus z over 2 factorial plus z squared over three factorial so on.

Now, we see that this series on the right hand side seems to have no difficulties at z equal to 0. On the other hand, if you blindly try to compute the value on the left hand side we run into difficulties, but so, if we assign the value of this function to be exactly the limit to which this series expansion of the right hand side is going to at z equal to 0.

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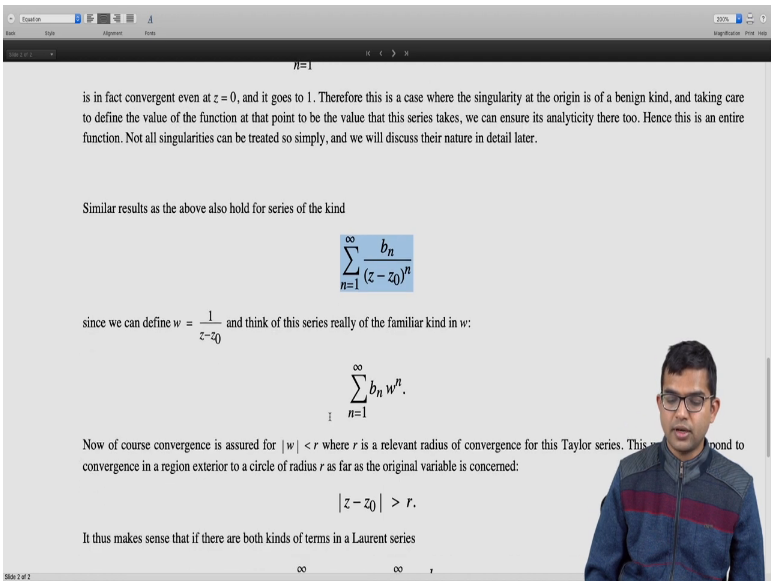

Then basically there is no issue. We have cured the singularity. And you know this series in fact, you sort of define this value of this function at that point using this series itself, right. So, this series and then we argue that if you have a Taylor series of this form or you know so, in this case it is you know it starts from 1. So, it's a constant term plus z over 2 factorial so on.

If you have a Taylor series or a Maclaurin series you know that is the value of this function at that point, right. So, we have just seen that this function is going to be convergent at all points interior to this you know the circle of interior to the region defined by this radius of convergence, ok.

So, the essential idea is that there are these kinds of singularities which can be cured just by using the series expansion and then arguing that the series is convergent there. So, there is a well defined limit and so, you can actually take a derivative which is analytic. All the nice properties of analyticity will hold at every point which can be checked, ok.

So, similar results will hold also for Laurent series expansions, right. So far we considered Taylor series expansions, but in fact, if you have a series of this kind where you have powers in 1 over z minus z naught, those two have similar results. So, the reason is that you can actually think of 1 over z minus z naught as some other complex number w.

And so, basically you are working with a series of this kind b n w to n, where you know n goes from 1 to infinity and so, this is really like a Taylor series in w and so, then we would argue that this has its own circle of convergence, so, region of convergence and it has a certain radius of convergence which. Suppose we find out that its radius of convergence is small r then we are assured that this series is going to be convergent whenever mod of w is less than r.

But what is meant by mod of w being less than r? It is the same as saying mod of 1 over z minus z naught is less than r which is the same as saying mod of z minus z naught is greater than r.

So, that is why, so, whatever you know results seem to hold and inside the red region of convergence or in the interior of a circle of convergence a similar type of story holds for you know series of this kind except now you must say that the region is exterior to this circle.

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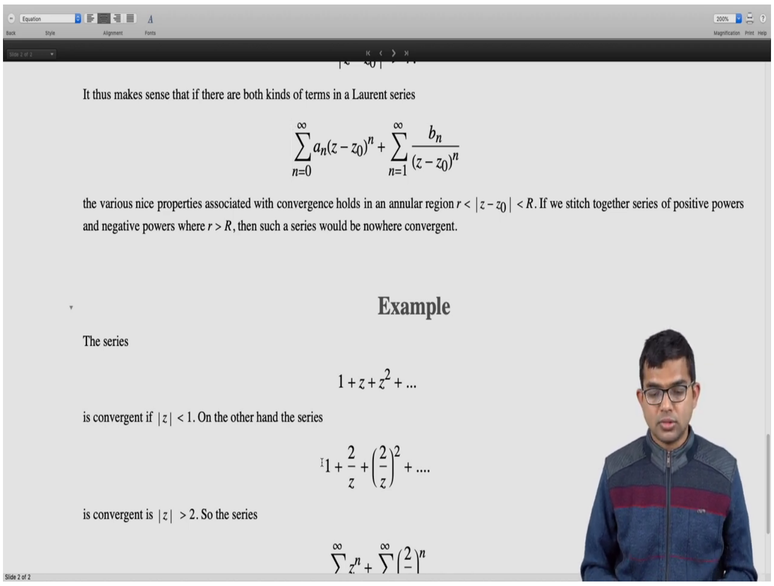

And so in fact, if you stitch together terms of this kind if you and then you make a Laurent series. You take terms of this kind and then at terms of this kind and then there is a region of convergence for this series. So, mod of z minus z naught must be less than capital R as far as this series is concerned if it has to be convergent r must be less than mod of z minus z naught. So, mod of z minus z naught must lie outside small r.

Now, therefore, if you want to have an annular region well defined annular region then it better be that the small r is less than capital R, otherwise if it happens that small r is greater than r capital R then such a series would be nowhere convergent. If you artificially try to bring together you know one series of this kind and another series of this kind and try to make an overall series you may actually end up with a as an overall series Laurent series which is not convergent anywhere.

So, let us quickly look at one example of this kind. So, we know that this series 1 plus z plus z squared so on is convergent whenever mod z is less than 1. On the other hand the series 1 plus 2 over z plus 2 over z whole squared so on is convergent whenever mod of 2 by z is less than 1, in other words mod z is greater than 2, right. So, that means, the small r in this case is 2 and large R in this case is actually 1.

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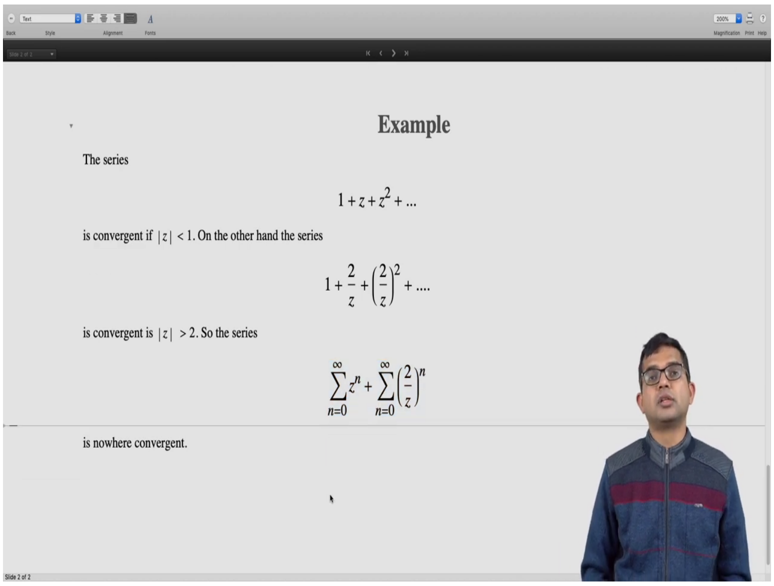

So, there is no region where you know if you just add these two series together there is nowhere that simultaneously both of these will converge. So, overall this is a Laurent series which has no region of convergence, ok. So, that is all for this lecture.

Thank you.