Mathematical Methods 2

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Complex Variables
Lecture - 36
Residues

Ok. So, we have seen how isolated singularities are very important. And so, for isolated singularities there is a Laurent series expansion available, and among all the coefficients that come up when you have a Laurent series expansion there is one coefficient which is of great importance and that is called the Residue, right. So, in this lecture, we will take a closer look at the residue and how to compute residue considering several examples, ok.

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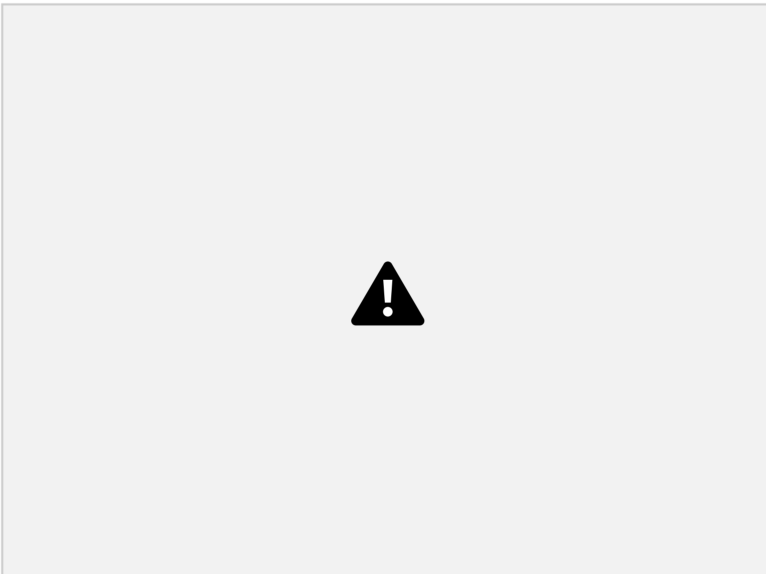

So, the Laurent expansion looks something like this. So, there is the regular part a naught plus a 1 times z minus z naught plus, so on. And then, there is the singularity part b 1 over z minus z naught, b 2 over z minus z naught, the whole square and so on, right. Among all of these coefficients b 1 occupies a central spotlight, right. So, it has this very important value associated with it, right, and that is called the residue, right.

So, let us look at how we can come up with sort of an algorithm to work out the residue for you know function given some singularity isolated singularity. So, in particular, our discussion will be to identify or to work out these residues at poles, right. Poles are the most frequently encountered kind of singularities. So, essential singularities are sort of pathological, and again removable singularities are removable, therefore, they are not so exciting.

On the other hand, poles appear repeatedly and there also there is a prescription that we can write down. And that is what we will do in this lecture. So, one way to identify a pole is to keep on finding successive limits of this kind.

So, you just multiply by z minus z naught and you know f of z, you have identified it as an isolated singularity. So, you multiply by z minus z naught and then try to evaluate this element z going to z naught. And if you find that it is still this limit does not exist, then try to find z minus z naught squared times f of z, take the limit z minus z naught. Even if this limit does not exist, go to the next order and so on.

The smallest integer m for which this limit is finite would be the order of the pole, right. So, this is often; so, this kind of a sort of trial and error method is often you know quite a practical way to work out the order of a pole. So, having identified this pole, how do you work out the residue, right. So, these are the two aspects which we would often have to worry about and find answers to. One is what is the order of a pole and the second is what is the residue at that point, right.

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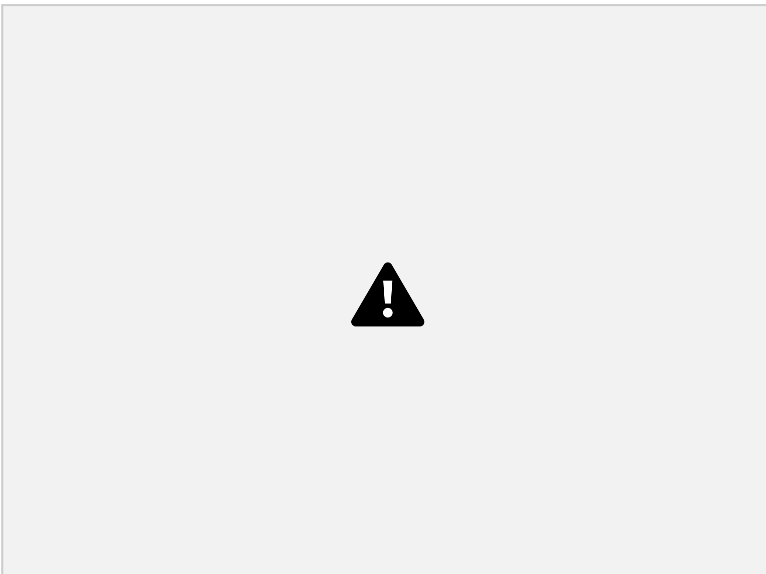

So, to find the residue, we observe that if you multiply this function f of z by z minus z naught the whole power m, right. So, since it is a residue of order it is a pole of order m, which means that there are no terms beyond b m. So, there is no b m plus 1, so on. So, it ends at that point b m. So, if you multiply this function by z minus z naught the whole power m, so indeed the entire thing becomes regular, right, as we have seen.

So, this is, so this is an analytic function. So, basically it is a Taylor expansion as far as this function is concerned. And so, now, this is a very nice smooth function, an analytic function you have. And from this function can you what can you do to this function to pull out this coefficient b 1? It is b 1 which we care about. That is the residue.

So, the way to find b 1 is of course, to successively differentiate this function m minus 1 times. So, if you differentiate it once and then this b m will go away, and if you differentiate it a second time b m minus 1 will go away, and the third time b m minus 2, so on, we will keep on going away. And then, if you successively differentiate m minus 1 times, you will be left with just this constant b 1, but with this factor which is m minus 1 the whole factorial, right.

And all these higher order terms you can put them to 0 by just taking the limit z going to z naught. Because all of these terms which appear before b 1 will have some at least one factor of z minus z naught. So, if you put z equal to z naught all of them will vanish. And as far as this constant, so b 1 will be just a constant, but attached along with it will be m minus 1 factorial.

So, therefore, the prescription for finding the residue of an nth order pole is to do just this. Take the m minus 1th derivative and you know at this point, take the value of this function at z equal to z naught and then divide by m minus 1 factorial and you are done, right. So, this is an easy enough prescription to write down. Not necessarily always the easiest way to work it out, but there is a ready prescription available, right.

So, let us look at some examples where we actually compute these residues, ok. So, the first example is you know of a function of this kind of very simple z plus 1 divided by z square plus 9. So, the denominator is basically z plus 3 i into z minus 3 i.

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So, there are two residue two poles, one of them is at 3 i and the other one is at minus 3 i. And both of these are evidently simple poles because if you simply multiply by z plus 3 i or by z minus 3 i that pole is gone, right. So, it's a simple pole, each of them. So, and the residues at these points are simply given by if you want to evaluate it at z equal to 3 i. So, you just multiply by z minus 3 i and then put the value z equal to 3 i and you are done. So, that gives you the residue to be half minus 1 by 6 i.

And on the other hand, if you want to evaluate the residue at the point z equal to minus 3 i, you multiply by z plus 3 i and put the value z equal to minus 3 i and this gives for you a residue of half plus 1 by 6 i, right a straight forward example.

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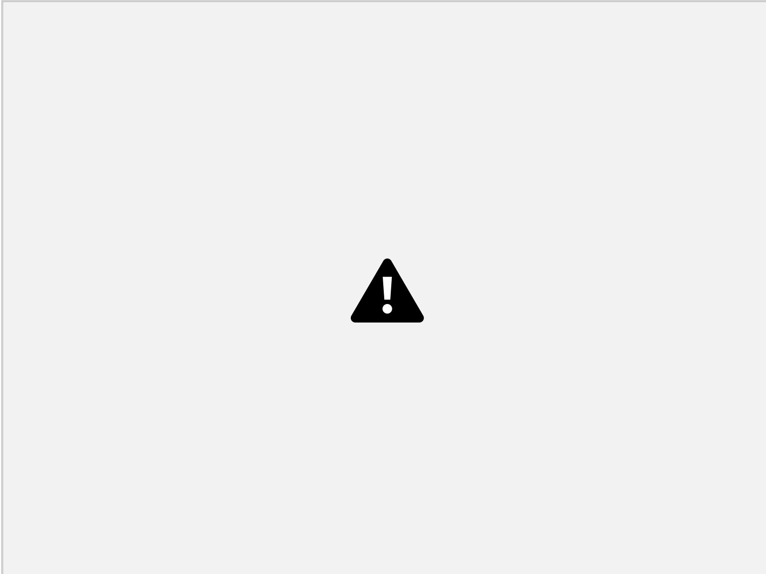

So, look at another example. So, if you consider a function like this z cube plus 2 z, so you have some polynomial sitting in the numerator and z minus i, the whole cube in the denominator, right. So, it is certainly a pole, right because you know if you put z equal to i, you immediately see that there is a singularity there and it is, of course, isolated.

And you know if you have a scenario like this, then it is useful to just define a function phi z. And so, you think of f of z as some function phi of z divided by z minus i the whole cube, and where phi of z is z cubed plus 2 z, is just a polynomial, so its entire. And phi of i is nonzero, so it is not over 0 by 0 form, right. So, then we conclude that indeed this function has a pole of order 3 at z equal to i and the residue is given by this precipitation we have.

So, you have to take the second order derivative, second derivative of this numerator phi of z, right, phi of z is of course, obtained by multiplying f of z by z minus i the whole cube, right. So, that is what you are supposed to do which is to multiply by this factor which is the power is basically the order of the pole. So, which is already done, phi of z.

So, if you take the second derivative of this, you get 3 z squared then 6 z and then you have to evaluate this at the point I, so you get 6 i divided by 2 factorial which is, so the answer is just 3 i, right. So, this is the residue of this function at the point z equal i.

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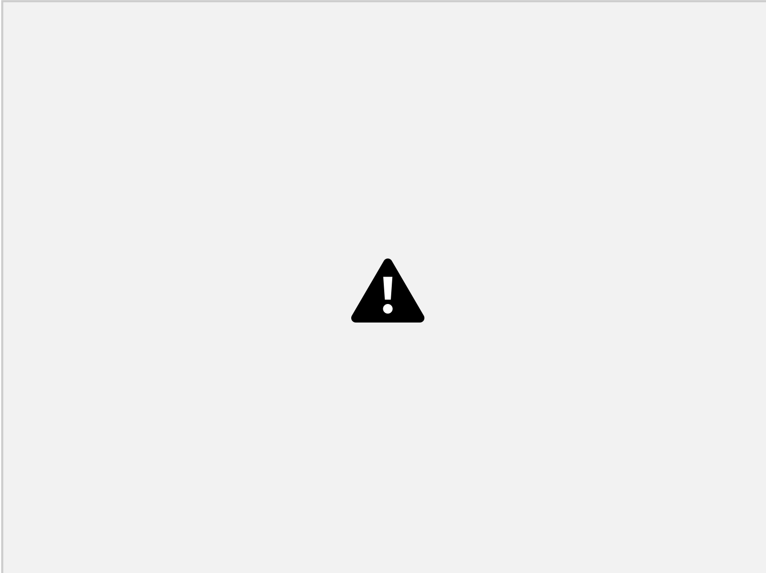

Now, if you consider this function 1 over z times e to the z minus 1, so oftentimes it is simplest to actually just find the Laurent expansion of this function which in turn can be actually got from the Taylor expansion of the denominator. If you just simply do that you can work out the order of the pole and also to work out the residue, right.

So, in this case for example, you see that the function f of z can be written as one over z times this e to the z, you know this 1 goes away, and so, you are left with just z over 1 factorial plus z squared plus you have to divided by 2 factorial plus z cube by 3 factorial plus, so on. And then, you can actually pull out this 1 over z from the second of these functions and then, so you get 1 over z squared times a function which clearly does not have a singularity at the point z equal to 0.

So, basically you have a singularity at z equal to 0, then you are able to pull out this 1 over z from this second function, 1 over e to the z minus 1. And so, therefore, it is evident that this has a pole order 2, because of this other function which you can call phi of z. So, phi you can think of f of z as phi of z divided by z squared where you only know this function phi of z in terms of this complicated looking expansion, but it is evident that there is no pole for this function at z equal to 0, right.

So, if you, which you can see by checking it by directly putting z equal to 0 it will just be 1. So, this function phi of z is 1 over 1 plus z by 2 of 2 factorial plus z squared by 3 factorial, so on. So, now this function you can take a derivative of this, right. So, we go back to our description.

So, you have to, since it is a pole of order 2 you have to take a derivative of this function and then put z equal to 0. So, the way to take a derivative of this complicated looking function is to use the quotient rule. So, if you have 1 over some function what you do is u over v is v d u minus u d v.

So, in this case v d u will be just 0, so it is just minus u d v, so which is minus minus of 1 over you know this stuff times the derivative of the denominator. So, the derivative of the denominator will be just 1 over 2 factorial plus 2 z by 3 factorial plus 3 squared by 4 factorial, so on. And the denominator has to be square, and you have to take the value of this at the point z equal to 0.

So, you see that the denominator will just get reduced to 1 and you take this limit, and then the numerator is going to give you just the minus half. So, that is the residue of this function at the point z equal to 0.

So, there are other ways of evaluating the residue for this function and we can cross check it. But basically, this is a direct way. You just simply expand, find the Taylor series expansion of the denominator and then you know argue from first principles that the residue for this function is just minus half.

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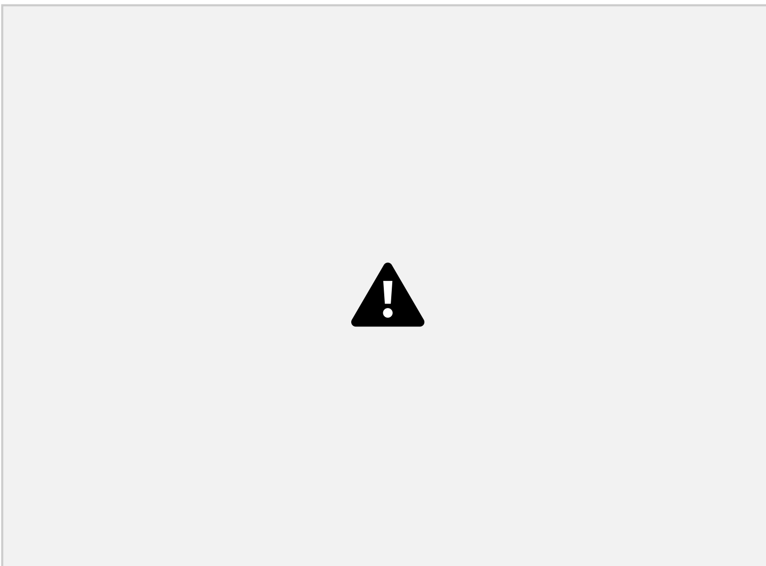

Now, zeros of analytic functions are you know intimately connected to these poles, right. So, in fact, if you take if you look at functions which are all the form numerator divided by the denominator and if the denominator has 0 of a certain order, then that will lead to a pole of the same order provided the numerator has no mess in it, right. So, let us define this carefully.

A function f of z that is analytic at a point z naught is said to have a 0 of order m and z naught ok. First of all, f of z naught must be 0 and it is not just 0, but it's a 0 of order m. So, what that means, is that you will be able to write down f of z as z minus z naught to the power m times g of z, where g of z is an analytic function which does not become 0 at the point z equal to 0, right.

So, that is important. So, g of z is analytic at z naught and it is also nonzero at z naught. And so, you should be able to sort of plot this factor z minus z naught the whole power m. And so, then you say that it is a 0 of order m, right. So, for example, this polynomial z squared minus 4, you can write it as z plus 2 times z minus 2. So, it has 0s at plus 2 and at minus 2 each of order 1, right. So, you know the definition is very analogous to the kind of definition we have for the order of a pole, ok.

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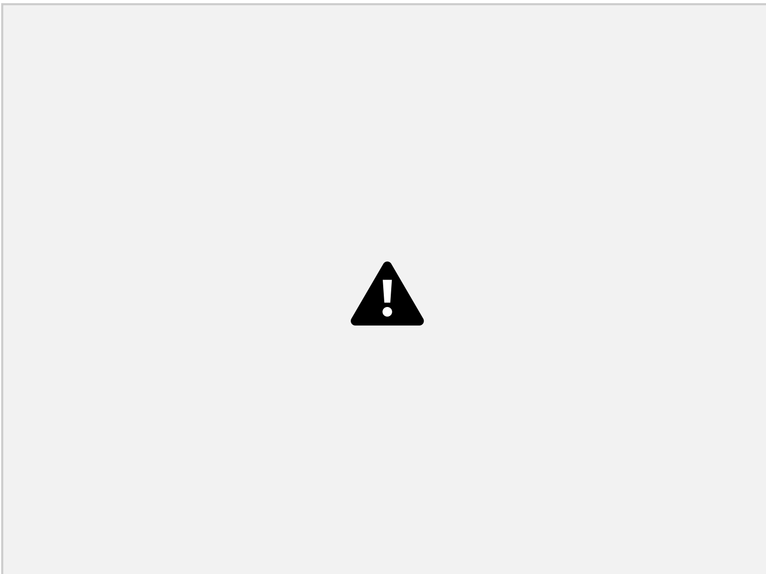

So, if you have a function like z times sin of z, it has a 0 of order 2 at the origin because z will contribute one order and sin of z contributes the other. So, another way to think of this is to just find the value of the function and find the value of the derivative at that point if. So, f of z, 0 is 0 f prime of 0 is also 0 and, but f double prime of 0 is nonzero. So, find the first derivative at that point which is nonzero, and so, that will be the order of the 0 plus 1, right.

So, that is also another way of arriving at the same answer for the order of a 0. So, we can express this function as f of z is equal to z squared times g of z, but g of z is you know you have to be careful about how you define this g of z, right. So, this is also a way of defining this function where the singularity has been removed, right. We have seen this. So, g of z must be defined as sin z over z for all z naught equal to 0 and it is defined as 1 at z equal to 0.

It is an analytic function. We have seen this. And it is nonzero at the point as z equal to 0. It is explicitly non-zero and you know we have pulled out a z square just like how the definition for the order of you know 0s was given. Now, so, why do we care about such 0s of function because of their connection to poles, right. So, we have this couple of results which are very useful.

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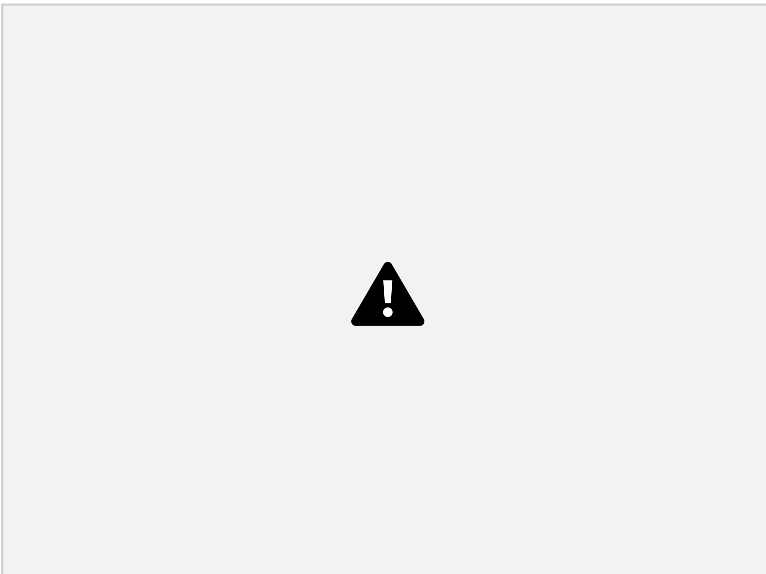

So, one of them is that if you have two functions p of z and q of z and both are analytic at a point z naught, such that p of z naught is not 0. So, the numerator does not have a 0 at that point whereas, the denominator has a 0, right. So, the denominator has a 0 at a certain point, then of course, you have a pole at that point, right. It has a 0 of order m. So, then the statement is that the quotient function f of z is equal to p of z divided by q of z is going to have a pole of order m at that point, right.

So, it seems straightforward enough. It is possible to argue for this a little more rigorously, but I think it is quite intuitive that if you have a function p of z which is not 0 at this point, and the denominator q of z has a 0 of a certain order then the overall function f of z has a pole of order m at that point z naught. So, let us look at an example.

So, if you take two functions like p of z and q of z. One of them here, if p of z is just 1, it is an entire function: q of z is z times e to the z minus 1. Now, both of them are actually entire, and so, you should divide one by the other. So, 1 over z times e to the z minus 1. Then, I mean the pole that is there at the origin can be extracted from the order of the 0 of the denominator, right.

So, we have already seen that z times e to the; well I mean we can argue that z times e to the z minus 1 has a 0 of order 2. Just like we argued that z times sin z has a 0 of order 2 at the origin. So, likewise z times e to this z minus 1 also has a 0 order 2 at the origin. Therefore, this function 1 over z times e to the z minus 1 has a pole of order 2. As we have already seen, we work this out from first principles, but we can also argue based on this result.

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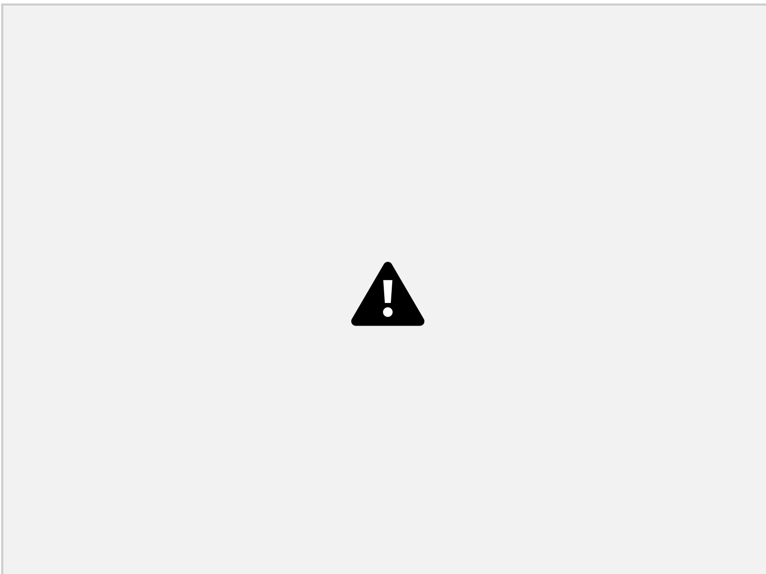

So, there is another result which is very useful and that is that p of z and q of z, if they are two functions that are both analytic at a point z naught, such that p of z naught is not 0 and q naught q of z naught is 0. So, the denominator is 0, the numerator is not 0, and if you are able to find that q prime of z naught is not 0, right. So, the derivative of the denominator is not 0, then you are guaranteed that z naught is a simple pole of the quotient function.

So, now you take this quotient function p of z divided by q of z and then you can simply write down the residue also straight away. And so, the residue is simply given by p of z naught divided by q prime of z naught, right. So, if you are able to be sure that this function f of z has a simple pole, then you simply work out p of z naught and then divided by q prime of z naught and you are done, right.

So, the reason this works out like here is because after all residue of this function f of z at z equal to z naught is basically residual of this function p of z divided by z minus z naught times g of z. Since, the denominator has a you know a pole of order, the denominator has a 0 of order 1, the reason is because q prime of z naught is not 0.

So, that immediately means that you will be able to write down the denominator as z minus z naught times g of z. And so, now, it is a matter of simply multiplying this function by z minus z naught and then you just have to take the limit of z going to z naught, right. But g of z naught is actually nothing, but q prime of z naught, right. So, this is also something which directly comes from you know the definition of you know this 0 of order 1, right.

After all you are given that you have a function q of z and it has a 0. So, you are able to pull out z minus z naught and then you are also given that q prime is not 0, therefore, indeed g of z naught is nothing, but q prime z naught. So, that is all. So, this is a straightforward result. And it is useful to just directly invoke this result and to work out residues.

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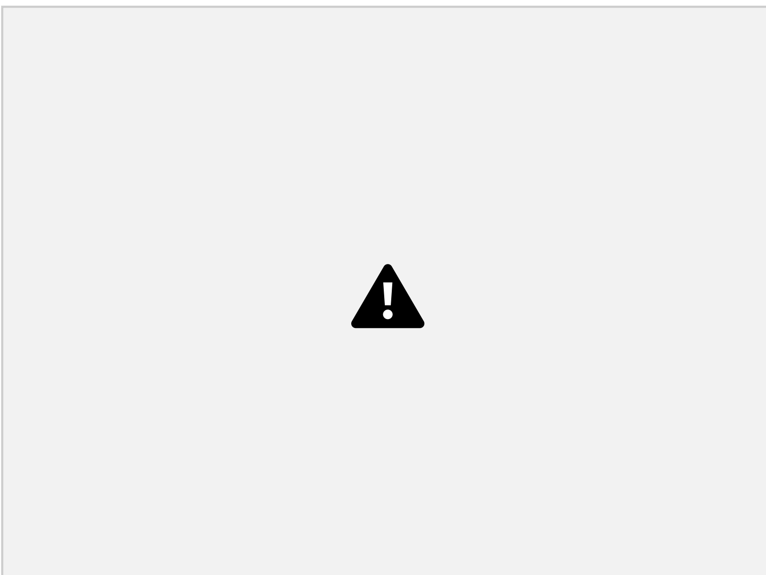

For example, if you have a function like cot z cos z divided by sin z. Now, sin z has a bunch of 0s which are sitting on the real axis. So, they are located at n pi. And all of these are isolated singularities and they are all 0s, I mean they are 0s of sin z and singularities of cot z. And so, the way to work out you know using this previous result, the way to work out the residue is to first find out p of n phi which is minus 1 to the n, right.

So, you have to just put cosine of n pi which will be minus 1 to the n which is not 0, right. So, that first condition is met. q of n pi is 0, so is after all it is a 0 at that point n pi is a 0 of the function sine z and q prime of n pi which is cosine of n pi is again minus 1 to the n which is not 0. So, each of the singularities is equal to n pi is a simple pole and the residual is simply given by p of n pi divided by q prime of n pi which are actually both the same in this particular case.

And so, therefore, the residue is just 1, right. So, where we have directly invoked, this is you could also work this out from first principles.

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Look at, let us look at just one more example. And so, if you have tan of z divided by z squared. Now, you can write this as sine of z divided by z square times cosine of z, suppose we focus on the point z equal to pi by 2, right. So, the singularity here comes because the cosine of pi by 2 is 0. So, z equal to pi by 2 is a 0 of the cosine function.

So, sine of z and z squared are well-behaved at this point. So, you can actually define this function f of z to be p of z over q of z, where p of z sin z divided by z squared. So, since you are at z equal to pi by 2, there is no problem with this function sin z by z squared. It would be problematic, if you are looking at it at the origin. And q of z is equal to cosine z. So, the mess really comes from the cosine of z which has a 0 at the point pi by 2.

Now, since p of pi by 2 is simply sin of pi by 2 by z squared, sin of pi by 2 is 1, z squared is pi squared by 4, so it is 4 by pi squared which is nonzero; q of pi by 2 is 0, q prime of pi by 2 is minus 1, and it is minus sign of z and at pi by 2 minus 1 which is nonzero.

The singularity at z equal to pi by 2 is a simple pole and whose residue is given by you know just this result that we have written down; p of pi by 2 divided by q prime of pi by 2 which is 4 by pi square divided by minus 1. So, the answer is minus 4 by pi squared.

So, why do we care so much about the residue, this is something that we will discuss in the you know upcoming lectures. But in this lecture, we saw a bunch of different you know methods for evaluating these residues of functions whenever they have any singularity or an isolated singularity and in particular for poles of order m, ok.

Thank you.