Mathematical Methods 2
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Complex Variables
Lecture - 40
Analytic Continuation

So in this lecture, we are going to discuss the notion of Analytic Continuation and this is going to bring us to the close of our discussion on Complex Variables and Applications. And so, before we move on to the next topic it's useful to briefly discuss a very beautiful idea and that is the idea of analytic continuation, ok.

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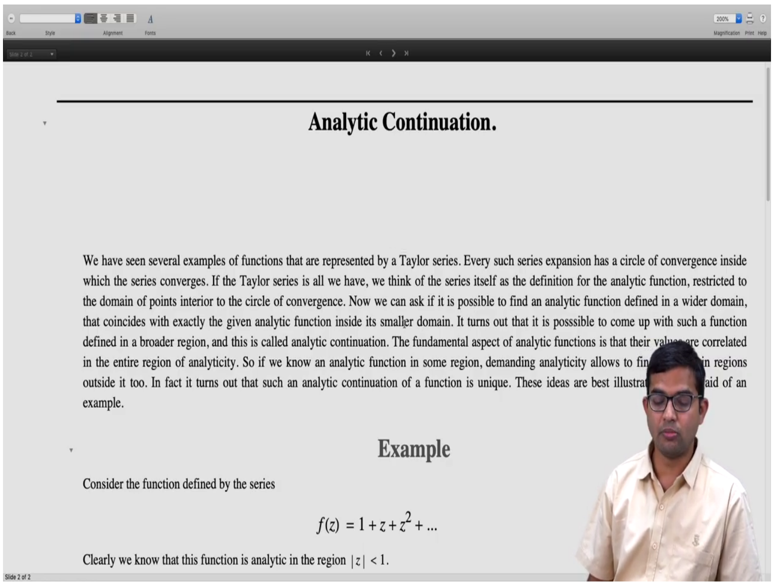

So, we have seen several examples, where you know analyticity you know impose many constraints on a function; a function which is analytic somehow seems to sense its value far away from where it is, right. If you know something about a function at a certain point or in some region, then you can say somehow that this function takes far away, right.

So, we have seen for example, that analytic functions have derivatives to all orders. We have seen how the Cauchy integral formula also suggests this kind of underlying correlation between the value of a function at a point and far away from it as well, right.

So, in this lecture, we will look at what is called analytic continuation. Suppose, you have a function which is defined in terms of a series and Taylor series as we have seen has a region of convergence, there is a circle of convergence inside which this Taylor series is convergent and so that function is well defined inside this circle of convergence.

Now the question is, is it possible to find a function which is well defined in a broader region and which still agrees with the value that the function takes in the region where it is already defined and such that it is analytic everywhere, right? So, the answer turns out to be yes.

It is possible to do this analytic continuation and in fact there is a unique way to you know continue analytically, right. So, we are not going to go into the specifics of or the, you know detailed way of stating these theorems. But I mean basically the idea is we will illustrate this idea of analytic coordination with the help of a rather familiar example, right.

So, consider the series 1 plus z plus z squared so on, you know it is going all the way up to infinity and we know that this Taylor series is convergent when mod z is less than 1.

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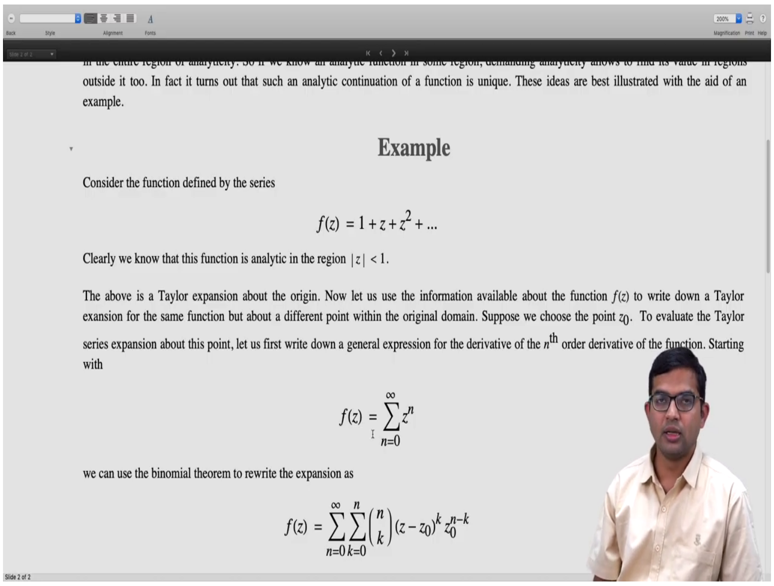

Now, suppose I mean this is the information we have, using this we can construct a meaningful Taylor series about some other point, right. Suppose we choose some point z naught and in order to evaluate the Taylor series expansion about this point. So, let us first write down a general expression, right.

So, we start with this expression for the function f of z, but then we want to be able to rewrite this as you know in place of z to the n, we want to write it as z; you can think of z to the n itself as being expanded about the point z naught, right. So, that is going to be basically a binomial expansion, where you know k goes from 0 to n and n choose k z minus z naught the whole to the k z naught to the n minus k, right.

So, this is just simply writing z to the n as z minus z naught plus z naught the whole power n and then I have just used the binomial expansion.

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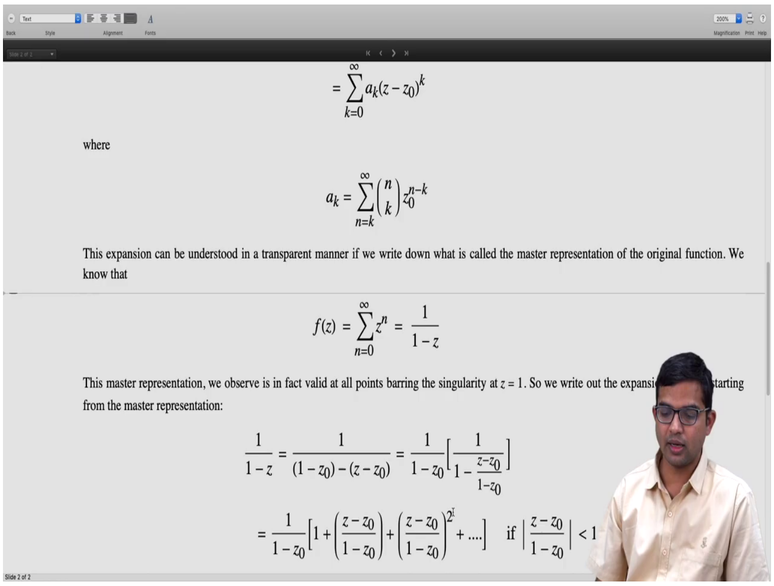

So, now basically if you look at this, this looks like summation over k going from 0 to infinity a k times z minus z naught to the whole k. Now, where a k is given by you know summation over n which goes from k to infinity and choose k z naught to n minus k, right.

I am just using the fact that, you know I am collecting all these terms where powers of z minus z naught to the k come up. And if I choose a particular k, so that is a k and that is going to actually involve an infinity somewhere; k goes all the way where n goes from k to infinity, right. So, you can check this.

So, basically the point is that, there is a way to convert a Taylor series about one point and rewrite it as a Taylor series about another one basically, right. Now, this whole thing can be understood in a transparent way, if we just use what is called a master representation of the original function.

So, although we write we started with this Taylor expansion, we can anyhow also know that in fact, this sums to 1 over 1 minus z, right. So, it is 1 over 1 minus z. And in fact if you look at this, although the left hand side; so this Taylor series is convergent only when mod z is less than 1. But on the other hand the right-hand side is actually a meaningful function, it is an analytic function at all points z except z is equal to 1.

So, there is a singularity at z equal to 1, but other than that it is actually an analytic function at any other point. So, we can in fact use this to come up with a Taylor expansion at about some other point z naught. Like whatever we are trying to do earlier you know this a k, we can work this out by rewriting this 1 over 1 minus z as 1 over 1 minus z naught minus z minus z naught, right.

So, we artificially sort of put this z naught in here and then we pull out this 1 over 1 minus z naught so, then we have to expand 1 over 1 minus z minus z naught divided by 1 minus z naught. So, then we argue that basically this is like the original series itself that we started, right. I mean as long as the mod of z minus z naught divided by mod of 1 minus z naught is less than 1, it is in fact valid for us to be to expand this as you know this kind of an expansion.

So, now this is valid in a different circle of convergence, right. Basically we are working with the same function really, but now the validity has changed to a different region, right.

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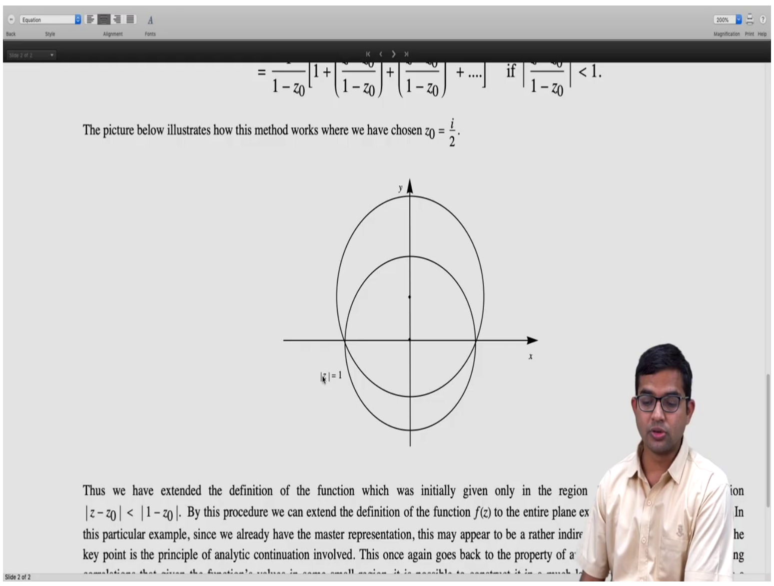

So, this picture below illustrates what has happened. So, initially we started with the circle of convergence which is basically the circle defined by mod z equal to 1. But then now if I were to expand about the same Taylor series itself, I know I expand about a different point or the same function itself if I expand about a different point, each validity is in a different circle.

And then I can go to another point in here and then expand about that point and then that is going to give me another circle; go to another point. Basically I can get a you know different Taylor series expansion about different points and this is going to be valid at all points, except of course there is one singularity, which is a genuine singularity and where it is non-analytic for sure, right. So, this function.

But the key point is that, we started with a function which was analytic in a region, but we managed to find an analytic function which is like a master function for this, which is valid in a much bigger domain. In this case it turns out to be the entire plane except that point, right. And this was entirely forced upon us just from our information that we had in a small region, right. So, this turns out to be quite a general result.

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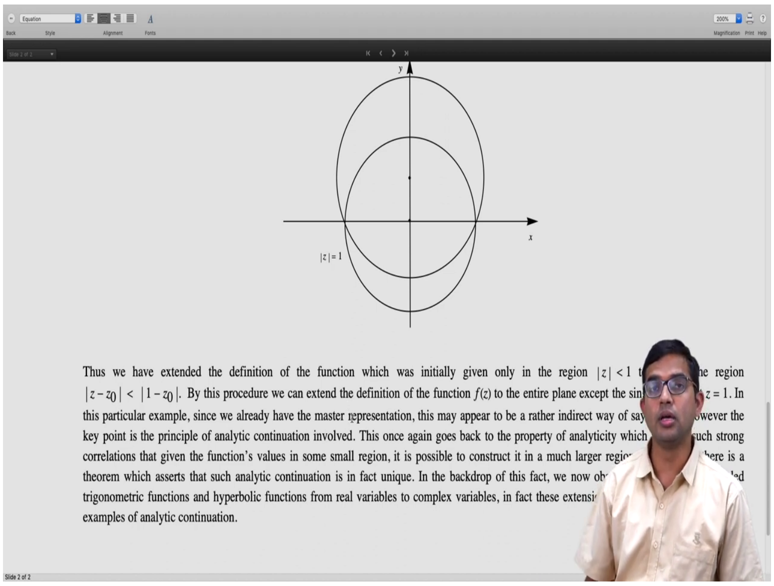

So in fact, this is something we have already done, when we generalize the idea of sin of x, right. So, we started with the sin of x as being defined only for real values of x. And then we said can we come up with a function which can take you know complex arguments and in such a way that you know sin of x and sin of z are the same whenever z becomes real, but the overall function that you have is analytic.

So, it turns out that there is a unique way of doing it, right. So, subject to certain conditions, you know what is a smaller domain that you are interested in and how you are expanding it, yeah extending it and so on. There are details about the precise statement of the theorem.

But basically the idea is that, you cannot get any other way of extending sin of x or e to the power x or any of these functions, hyperbolic functions all of these which we extended to the full complex plane. But it was not done in some arbitrary way, it was done in such a way that the function is analytic everywhere, right.

So, this is a powerful aspect of analytic functions. This would not happen for example, if you are working in you know functions of a real variable, you cannot you know, there is no unique way of extending your function to a bigger region such that you know the two functions overlapping your smaller region and then you impose some other nice properties; it is it does not work the way you know analytic continuation works for functions of a complex variable, ok.

So, with this we come to an end of our discussion of complex functions and applications of complex variables. There are more things one could have done and perhaps in a slightly more advanced course, which you know more details can be covered or you can use this material from here to build on this to study more advanced topics. But as far as we are concerned, we come to an end for you know in regards to this topic and then we move on to the next topic starting from the next lecture.

Thank you.