

**Mathematical Methods 2**  
**Prof. Auditya Sharma**  
**Department of Physics**  
**Indian Institute of Science Education and Research, Bhopal**

**Complex Variables**  
**Lecture - 40**  
**Analytic Continuation**

So in this lecture, we are going to discuss the notion of Analytic Continuation and this is going to bring us to the close of our discussion on Complex Variables and Applications. And so, before we move on to the next topic it's useful to briefly discuss a very beautiful idea and that is the idea of analytic continuation, ok.

(Refer Slide Time: 00:45)

**Analytic Continuation.**

We have seen several examples of functions that are represented by a Taylor series. Every such series expansion has a circle of convergence inside which the series converges. If the Taylor series is all we have, we think of the series itself as the definition for the analytic function, restricted to the domain of points interior to the circle of convergence. Now we can ask if it is possible to find an analytic function defined in a wider domain, that coincides with exactly the given analytic function inside its smaller domain. It turns out that it is possible to come up with such a function defined in a broader region, and this is called analytic continuation. The fundamental aspect of analytic functions is that their values are correlated in the entire region of analyticity. So if we know an analytic function in some region, demanding analyticity allows to find its values in regions outside it too. In fact it turns out that such an analytic continuation of a function is unique. These ideas are best illustrated with the aid of an example.

**Example**

Consider the function defined by the series

$$f(z) = 1 + z + z^2 + \dots$$

Clearly we know that this function is analytic in the region  $|z| < 1$ .

*(A video inset of Prof. Auditya Sharma is visible in the bottom right corner of the slide.)*

So, we have seen several examples, where you know analyticity you know impose many constraints on a function; a function which is analytic somehow seems to sense its value far away from where it is, right. If you know something about a function at a certain point or in some region, then you can say somehow that this function takes far away, right.

So, we have seen for example, that analytic functions have derivatives to all orders. We have seen how the Cauchy integral formula also suggests this kind of underlying correlation between the value of a function at a point and far away from it as well, right.

So, in this lecture, we will look at what is called analytic continuation. Suppose, you have a function which is defined in terms of a series and Taylor series as we have seen has a region of convergence, there is a circle of convergence inside which this Taylor series is convergent and so that function is well defined inside this circle of convergence.

Now the question is, is it possible to find a function which is well defined in a broader region and which still agrees with the value that the function takes in the region where it is already defined and such that it is analytic everywhere, right? So, the answer turns out to be yes.

It is possible to do this analytic continuation and in fact there is a unique way to you know continue analytically, right. So, we are not going to go into the specifics of or the, you know detailed way of stating these theorems. But I mean basically the idea is we will illustrate this idea of analytic continuation with the help of a rather familiar example, right.

So, consider the series 1 plus z plus z squared so on, you know it is going all the way up to infinity and we know that this Taylor series is convergent when mod z is less than 1.

(Refer Slide Time: 02:45)

...the entire region of analyticity... we know an analytic function in some region, obtaining analyticity allows to find its value in regions outside it too. In fact it turns out that such an analytic continuation of a function is unique. These ideas are best illustrated with the aid of an example.

### Example

Consider the function defined by the series

$$f(z) = 1 + z + z^2 + \dots$$

Clearly we know that this function is analytic in the region  $|z| < 1$ .

The above is a Taylor expansion about the origin. Now let us use the information available about the function  $f(z)$  to write down a Taylor expansion for the same function but about a different point within the original domain. Suppose we choose the point  $z_0$ . To evaluate the Taylor series expansion about this point, let us first write down a general expression for the derivative of the  $n^{\text{th}}$  order derivative of the function. Starting with

$$f(z) = \sum_{n=0}^{\infty} z^n$$

we can use the binomial theorem to rewrite the expansion as

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (z - z_0)^k z_0^{n-k}$$

Now, suppose I mean this is the information we have, using this we can construct a meaningful Taylor series about some other point, right. Suppose we choose some point  $z_0$  and in order to evaluate the Taylor series expansion about this point. So, let us first write down a general expression, right.

So, we start with this expression for the function  $f$  of  $z$ , but then we want to be able to rewrite this as you know in place of  $z$  to the  $n$ , we want to write it as  $z$ ; you can think of  $z$  to the  $n$  itself as being expanded about the point  $z_0$ , right. So, that is going to be basically a binomial expansion, where you know  $k$  goes from 0 to  $n$  and  $n$  choose  $k$   $z$  minus  $z_0$  to the whole to the  $k$   $z_0$  to the  $n$  minus  $k$ , right.

So, this is just simply writing  $z$  to the  $n$  as  $z$  minus  $z_0$  plus  $z_0$  to the whole power  $n$  and then I have just used the binomial expansion.

(Refer Slide Time: 03:45)

where

$$a_k = \sum_{n=k}^{\infty} \binom{n}{k} z_0^{n-k}$$

This expansion can be understood in a transparent manner if we write down what is called the master representation of the original function. We know that

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

This master representation, we observe is in fact valid at all points barring the singularity at  $z = 1$ . So we write out the expansion starting from the master representation:

$$\frac{1}{1-z} = \frac{1}{(1-z_0) - (z-z_0)} = \frac{1}{1-z_0} \left[ \frac{1}{1 - \frac{z-z_0}{1-z_0}} \right]$$

$$= \frac{1}{1-z_0} \left[ 1 + \left( \frac{z-z_0}{1-z_0} \right) + \left( \frac{z-z_0}{1-z_0} \right)^2 + \dots \right] \quad \text{if } \left| \frac{z-z_0}{1-z_0} \right| < 1$$

So, now basically if you look at this, this looks like summation over  $k$  going from 0 to infinity  $a_k$  times  $z$  minus  $z_0$  to the whole  $k$ . Now, where  $a_k$  is given by you know summation over  $n$  which goes from  $k$  to infinity and choose  $k$   $z_0$  to  $n$  minus  $k$ , right.

I am just using the fact that, you know I am collecting all these terms where powers of  $z$  minus  $z_0$  to the  $k$  come up. And if I choose a particular  $k$ , so that is a  $k$  and that is going to actually involve an infinity somewhere;  $k$  goes all the way where  $n$  goes from  $k$  to infinity, right. So, you can check this.

So, basically the point is that, there is a way to convert a Taylor series about one point and rewrite it as a Taylor series about another one basically, right. Now, this whole thing can be understood in a transparent way, if we just use what is called a master representation of the original function.

So, although we write we started with this Taylor expansion, we can anyhow also know that in fact, this sums to  $1$  over  $1$  minus  $z$ , right. So, it is  $1$  over  $1$  minus  $z$ . And in fact if you look at this, although the left hand side; so this Taylor series is convergent only when  $\text{mod } z$  is less than  $1$ . But on the other hand the right-hand side is actually a meaningful function, it is an analytic function at all points  $z$  except  $z$  is equal to  $1$ .

So, there is a singularity at  $z$  equal to  $1$ , but other than that it is actually an analytic function at any other point. So, we can in fact use this to come up with a Taylor expansion at about some other point  $z_0$ . Like whatever we are trying to do earlier you know this a  $k$ , we can work this out by rewriting this  $1$  over  $1$  minus  $z$  as  $1$  over  $1$  minus  $z_0$  minus  $z$  minus  $z_0$ , right.

So, we artificially sort of put this  $z_0$  in here and then we pull out this  $1$  over  $1$  minus  $z_0$  so, then we have to expand  $1$  over  $1$  minus  $z$  minus  $z_0$  divided by  $1$  minus  $z_0$ . So, then we argue that basically this is like the original series itself that we started, right. I mean as long as the  $\text{mod of } z$  minus  $z_0$  divided by  $\text{mod of } 1$  minus  $z_0$  is less than  $1$ , it is in fact valid for us to be to expand this as you know this kind of an expansion.

So, now this is valid in a different circle of convergence, right. Basically we are working with the same function really, but now the validity has changed to a different region, right.

(Refer Slide Time: 06:35)

$$= \frac{1}{1-z_0} \left[ 1 + \left( \frac{z-z_0}{1-z_0} \right) + \left( \frac{z-z_0}{1-z_0} \right)^2 + \dots \right] \quad \text{if } \left| \frac{z-z_0}{1-z_0} \right| < 1.$$

The picture below illustrates how this method works where we have chosen  $z_0 = \frac{i}{2}$ .

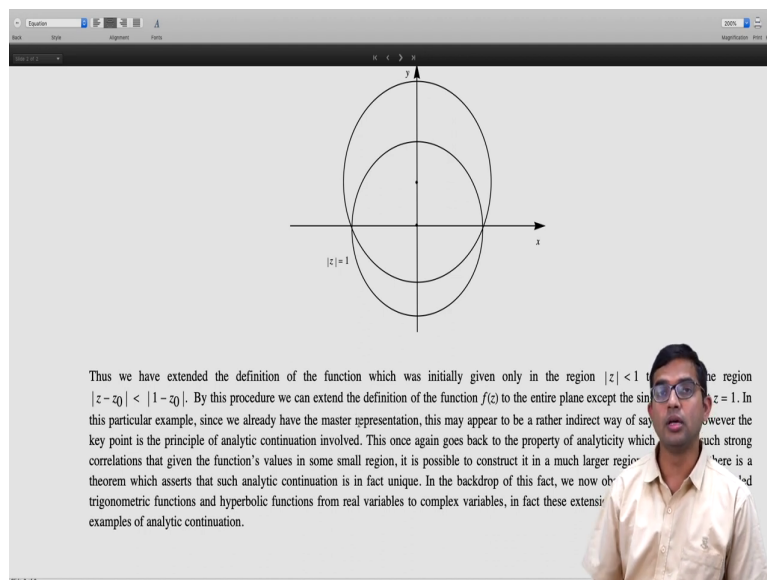
Thus we have extended the definition of the function which was initially given only in the region  $|z - z_0| < |1 - z_0|$ . By this procedure we can extend the definition of the function  $f(z)$  to the entire plane except at  $z = 1$ . In this particular example, since we already have the master representation, this may appear to be a rather irrelevant key point is the principle of analytic continuation involved. This once again goes back to the property of a function that since the function's value is given on a small region, it is possible to extend it to a much larger region.

So, this picture below illustrates what has happened. So, initially we started with the circle of convergence which is basically the circle defined by  $\text{mod } z$  equal to 1. But then now if I were to expand about the same Taylor series itself, I know I expand about a different point or the same function itself if I expand about a different point, each validity is in a different circle.

And then I can go to another point in here and then expand about that point and then that is going to give me another circle; go to another point. Basically I can get a you know different Taylor series expansion about different points and this is going to be valid at all points, except of course there is one singularity, which is a genuine singularity and where it is non-analytic for sure, right. So, this function.

But the key point is that, we started with a function which was analytic in a region, but we managed to find an analytic function which is like a master function for this, which is valid in a much bigger domain. In this case it turns out to be the entire plane except that point, right. And this was entirely forced upon us just from our information that we had in a small region, right. So, this turns out to be quite a general result.

(Refer Slide Time: 07:52)



Thus we have extended the definition of the function which was initially given only in the region  $|z| < 1$  to the region  $|z - z_0| < |1 - z_0|$ . By this procedure we can extend the definition of the function  $f(z)$  to the entire plane except the singularity at  $z = 1$ . In this particular example, since we already have the master representation, this may appear to be a rather indirect way of saying it. However the key point is the principle of analytic continuation involved. This once again goes back to the property of analyticity which allows such strong correlations that given the function's values in some small region, it is possible to construct it in a much larger region. There is a theorem which asserts that such analytic continuation is in fact unique. In the backdrop of this fact, we now observe how trigonometric functions and hyperbolic functions from real variables to complex variables, in fact these extend to complex variables. This is an example of analytic continuation.

So in fact, this is something we have already done, when we generalize the idea of  $\sin$  of  $x$ , right. So, we started with the  $\sin$  of  $x$  as being defined only for real values of  $x$ . And then we said can we come up with a function which can take you know complex arguments and in such a way that you know  $\sin$  of  $x$  and  $\sin$  of  $z$  are the same whenever  $z$  becomes real, but the overall function that you have is analytic.

So, it turns out that there is a unique way of doing it, right. So, subject to certain conditions, you know what is a smaller domain that you are interested in and how you are expanding it, yeah extending it and so on. There are details about the precise statement of the theorem.

But basically the idea is that, you cannot get any other way of extending  $\sin$  of  $x$  or  $e$  to the power  $x$  or any of these functions, hyperbolic functions all of these which we extended to the full complex plane. But it was not done in some arbitrary way, it was done in such a way that the function is analytic everywhere, right.

So, this is a powerful aspect of analytic functions. This would not happen for example, if you are working in you know functions of a real variable, you cannot you know, there is no unique way of extending your function to a bigger region such that you know the two functions overlapping your smaller region and then you impose some other nice properties; it is it does not work the way you know analytic continuation works for functions of a complex variable, ok.

So, with this we come to an end of our discussion of complex functions and applications of complex variables. There are more things one could have done and perhaps in a slightly more advanced course, which you know more details can be covered or you can use this material from here to build on this to study more advanced topics. But as far as we are concerned, we come to an end for you know in regards to this topic and then we move on to the next topic starting from the next lecture.

Thank you.