Mathematical Methods 2
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Orthogonal polynomials
Lecture - 42
How to construct orthogonal polynomials

So, we have seen how you know orthogonal polynomials lead to this vector space structure and some immediate consequences. So, in this lecture, we will start looking at some you know very general looking at a very general approach to construct such orthogonal polynomials. And so this general framework will come in handy when you look at many classes of such orthogonal polynomials.

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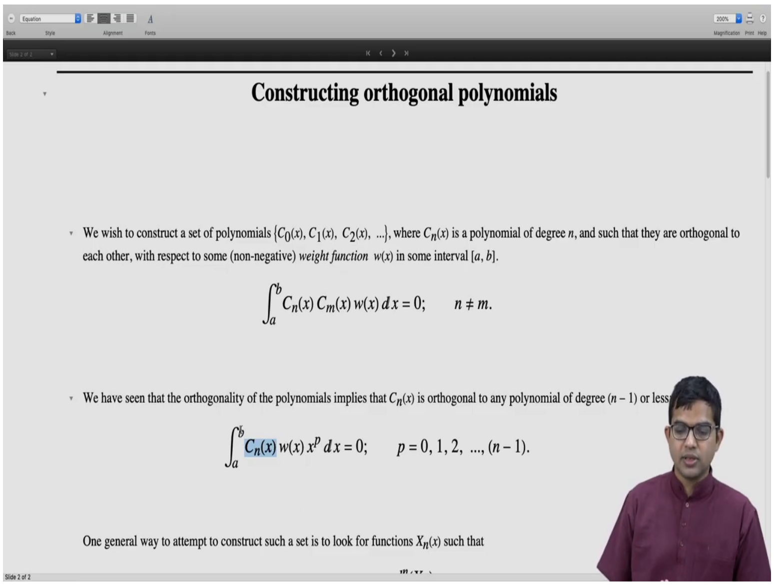

So, we have seen how you know we are looking at a bunch of polynomials C naught C 1 C 2 of successively increasing degree, and there is a weight function w of x and an interval a to b specified, and they are orthogonal right. So, the orthogonality is given by this relation whenever n is not equal to m.

Now, this orthogonality of these polynomials we have seen immediately implies that every such polynomial is orthogonal to any polynomial of degree n minus 1 or less, and specifically in fact, C n of x is orthogonal to every x to the p whenever p is less than n right. So, this relation holds. Now, if you want to construct a polynomial set of polynomials like this, we have to design this set in such a way that this relation works, this is a you know key relation right.

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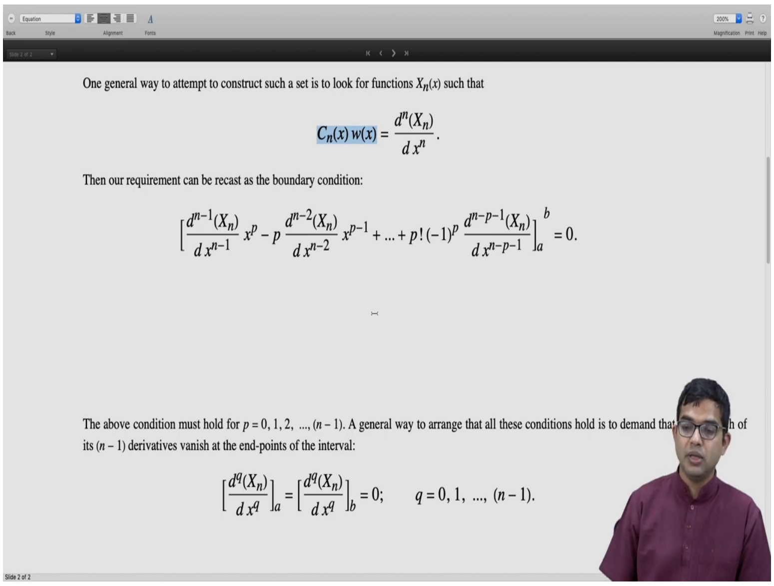

So, one way to try to do this is to construct you know these products such as C n of x times w of x to be you know the nth derivative of a function right. Suppose you do this, then what would happen? So, we would have you know we can invoke the integration by parts. So, we have you know this object is written as the nth derivative of this function, and so then what would we do.

So, we would say, so this is like u d v, well I mean this is u and this is d v. So, that can be written as u, v. So, in this case, it will be x to the p times d n n minus 1 divided by d X n minus 1 X n between you know a and b. So, those are the values you have to input and just evaluate the function at that point minus the derivative of this quantity p times x to the p minus 1. And then it is just as it is C n of x times w of x that will be just this guy.

Now, it is u v minus v v du. So, this is u, so that is right. So, it is going to be d to the n minus 1 by d x minus 1 right. So, this is something you have to convince yourself by spending a few minutes and putting it down on paper. So, the idea is ok, let us see.

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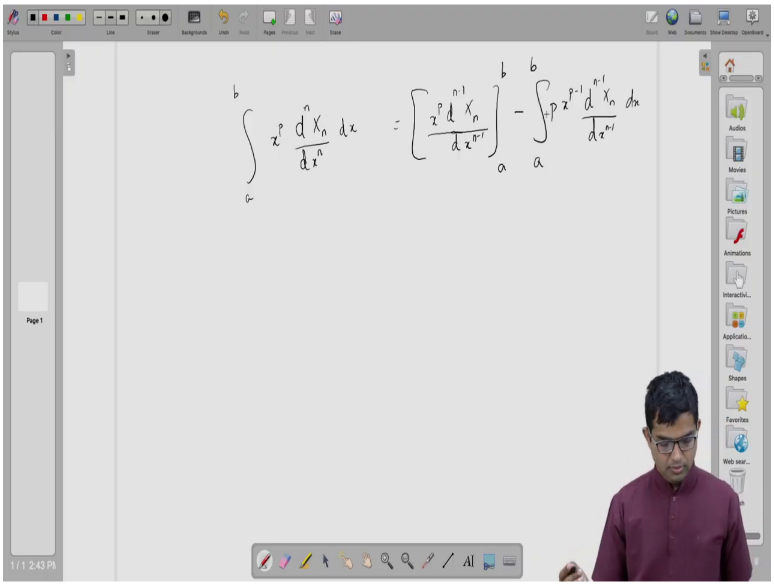

So, we have something like d to the n d to the n by d x to the n by dx to the n and this function on the left side is x to the p. So, we have something like this x to the p d x. So, this is the integral that we are doing. So, when you do this, so this is like this is u and this is v. So, this is so this is u d v. So, this we write it as x to the p into d to the n minus 1 x, well x n by d x to the n minus 1 evaluated at a and that b minus integral.

Now, it is an integral a to the a to b. Now, we have to take a derivative of this guy. So, it is going to be p times x to the p minus 1, and then we have to work with d to the n minus 1 X n by d x to the m minus 1 d x. But then again we can apply integration by parts to this new quantity; now there is a p sitting here. And the next time there is going to be a p times p minus 1, then p minus p times p minus 1 times p minus 2 and so on.

So, in the end, we will and also there are you know signs which keep changing alternating every time. So, you can convince yourself that indeed what this boils down to is you know just this, big sum right. So, there is a n minus 1 derivative of X n times x to the p minus p times the n minus 2th derivative of x n times x to the p minus 1 plus. So, on all the way up to p factorial times minus 1 to the p.

So, these signs keep alternating. And then you get a p factorial in n, so first time is minus p then you get a plus p into p minus 1 and so on. And finally, you have n the derivative that you have to take here is of the order n minus p minus 1 x X n, and you want this evaluated at b minus evaluated at a must go to 0 right. So, if you can somehow arrange for this, then this would work out right. So, for simplicity, a general way to arrange this is to actually just demand that all these derivatives are 0 at both the points at a and at b separately.

So, if you demand each of these quantities is 0, both at a and at b, then this gets automatically satisfied right. So, that is what we will do. So, suppose we make this as a condition that all these derivatives q equal to 0, q equal to 1, all the way up to n minus 1, they just simply vanish at both a and at b.

And automatically you know what we were looking for is satisfied right, so that is the approach we are going to take. And this is one way to construct a set of functions. At this point, you know we are looking at a bunch of functions C n of x such that they are orthogonal to x to the p where p is less than n right.

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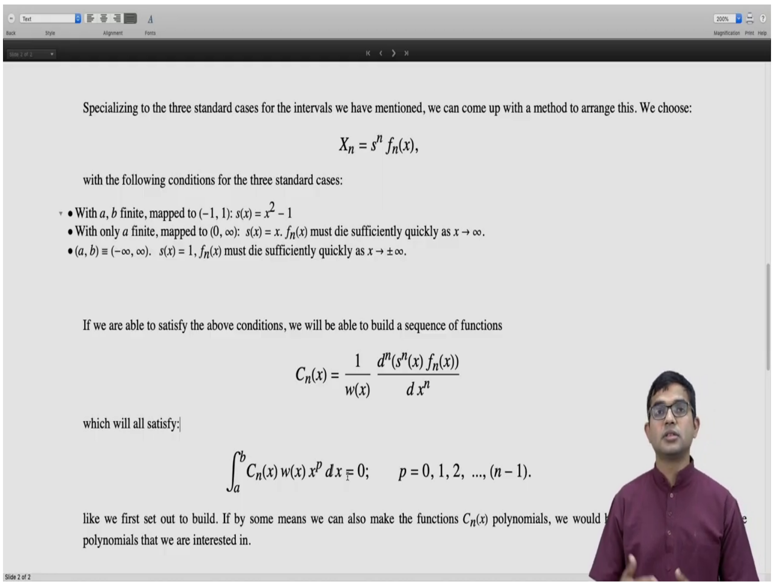

So, we have seen that there are these three special intervals right, all intervals can be recast as in one of these three forms, and so we would actually choose X n to take this particular form. It is convenient to write it as X n is equal to s to the n times f n of x break x n itself is a product of two functions x s to the n times f n of x. Now, this s is what is going to you know nicely take care of these endpoints right.

So, if a and b are taken to be minus 1 and plus 1, you choose s of x to be x squared minus 1, so x plus 1 times x minus 1. So, at both these ends this object is going to go to 0. And in fact you take it to the power n, so that you know all these higher order derivatives also go to 0 at both the end points.

And if with only a finite which then gets mapped in term to 0 to infinity, you take s of x to be x. Now, since you have x to the n you know you can take derivatives of this. And still when you put x equal to 0, there is going to be some power of x lingering around, so it is going to kill this function at the point x equal to 0.

But on the other hand, when you have the other limit to be plus infinity, the way to ensure that it falls off to 0 and all higher order derivatives follow to 0 is to make sure that this f n of x is going to die sufficiently quickly as x tends to infinity. And the third case is when s of x has no role you just put s of x to be 1. It is entirely in the hands of f n to ensure convergence, and so f n of x must die down sufficiently quickly at both ends right. So, these are the three scenarios we will consider.

And so if we you know plug this in and build the sequence of functions C n of x is equal to 1 over w of x nth derivative of s to the power n of x times f n of x, this is going to satisfy the condition that we want namely that orthogonality property between these functions, the sequence of functions with respect to x to the p where p is any integer positive or non-negative integer less than m right.

So, this is basically we are in the right direction. But at this point, these C n of x are just some functions we want to if you can somehow ensure that these functions are also polynomials then we would have come up with a prescription to generate such polynomials right.

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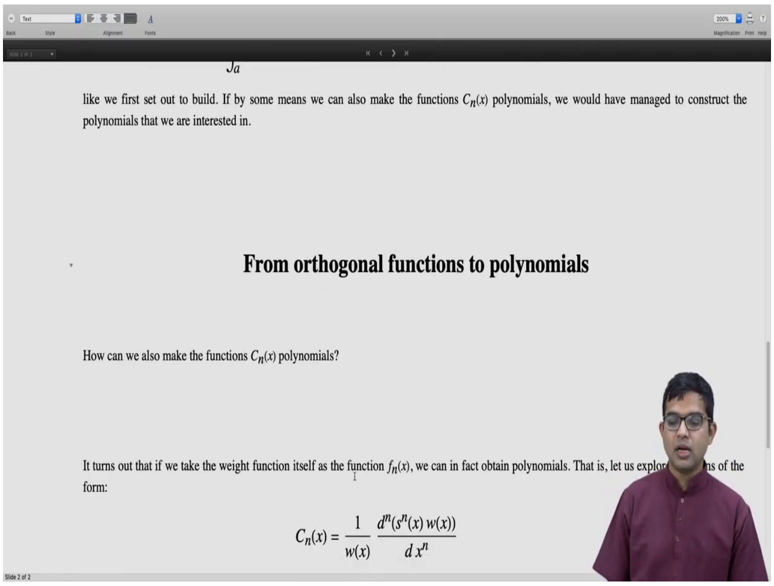

So, it turns out that you know a way to make these orthogonal functions, you know to become polynomials is to actually choose the weight function itself as this function f n of x right. So, at this point it is completely general, f n of x simply has to have certain nice properties far away right at the two ends.

But suppose you choose f n of x to be this weight function, and then it turns out right, so this is something it is kind of a trial and error approach. And then it so happens that we can actually show that if we choose these f n of x themselves to be w of x and demand certain properties of these w of x's, then we can generate a bunch of polynomials.

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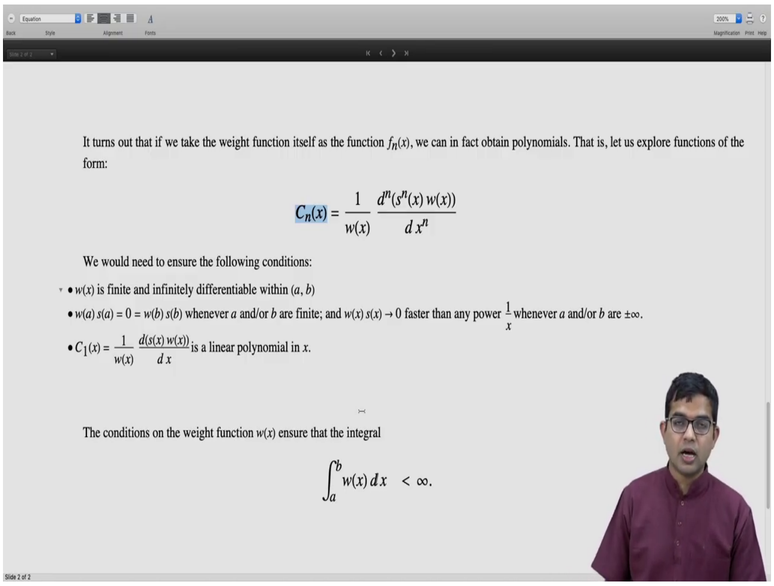

So, let us look at some of the conditions that you know will have to be satisfied. We will sort of state them, but in the following lecture and you know as we go along, we look at you know some of the details of this at this point we are sort of just stating how the what are the broad conditions which we need to put in right.

So, first of all we must choose w of x to be finite and infinitely differentiable within a and b right. So, w of x should be a really nice function because and it is also infinitely differentiable because after all these polynomials are defined in terms of these nth order derivatives. So, w of x must be sufficiently smooth that if you keep on differentiating it repeatedly you should get into no trouble right.

So, first of all w of x is finite and infinitely differentiable within this entire region of interest a to b also at the boundaries. So, w of a times s of a must go to 0, and at w of b times s of b must go to 0. And in the case of you know either a or b or both of them being infinite, then this w of x times s of x has to die down sufficiently rapidly. It should in fact go down faster than any power of 1 over x whenever a n and or b are plus or minus infinity right.

So, this ensures convergence of these you know various integrals involved, and in fact, a requirement you know an immediate consequence in fact of these conditions is that these weight functions are you know such that this integral a to b w of x dx is going to be finite right. So, your weight function cannot be some arbitrary function. Not only is it non-negative, but there is this interpretation of this weight function as a kind of measure right. So, it all adds up to some finite value right, so that is sort of inbuilt in these conditions.

And another, it turns out right. So, this we will see we will explore further some of these properties in the next lecture also. And following lectures is if you choose the first of these C 1 of x to be a linear polynomial, then automatically with these conditions, you can actually get all the following functions in this sequence to be also polynomials. We will just have to ensure that the first one is a polynomial, and that will actually give you the differential equation.

So, you work out this differential equation, solve for it. And then for a given s of x, w of x, you have chosen it in a nice way it is going to lead to the first polynomial. You force it to be a polynomial linear polynomial in x and that will automatically give you a whole sequence of polynomials that is going to be this orthogonal set of polynomials.

And we will see by playing with these weight functions and these intervals and these ss we can actually generate many classes of polynomials which many of which actually we are already sort of familiar with. But it is interesting that we have the sort of very general perspective from which we will come from the general to the particular right. So, this is the philosophy that we are adopting ok.

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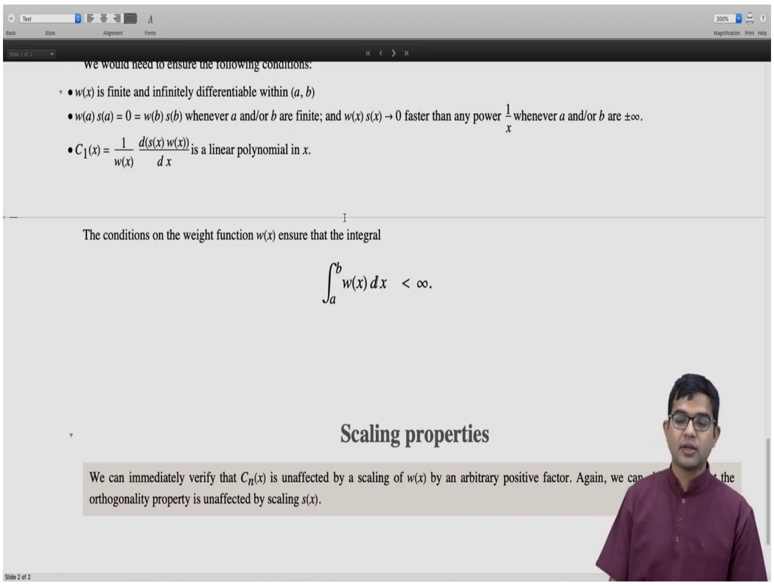

So, that is all for this lecture. But before we go off, I just want to very quickly mention that there are these very interesting scaling properties which we will also exploit right. One is that C n of x is unaffected by a scaling of w of x right C n of x is defined like this.

So, if you multiply w of x by some factor, since it is going to get divided here, C n of x is unchanged. If you scale w of x by an arbitrary positive factor positive factor because after all w of x has to be a non-negative function. And again we can also check that the orthogonality property is unaffected by scaling s of x right.

So, the orthogonality property you know comes way back here. So, this is something that you can check right. So, s of x comes here. If you scale s of x, you will still get a bunch of you know orthogonal functions.

So, the orthogonality property is unaffected by scaling s of x. So, this is also something that you should check explicitly and convince yourself that this holds right. So, these properties will turn out to be useful in our future discussion. That is all for this lecture.

Thank you.