Mathematical Methods 2

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Orthogonal Polynomials
Lecture - 51
The generating function corresponding to Legendre polynomials

So, we continue our discussion of Legendre polynomials. In this lecture, we will look at the generating function which is associated with Legendre polynomials, and briefly sketch how it can be used to derive some of the results which we have already seen in a very clever way, but also it opens up possibilities of deriving other results which some of which will be part of homework ok.

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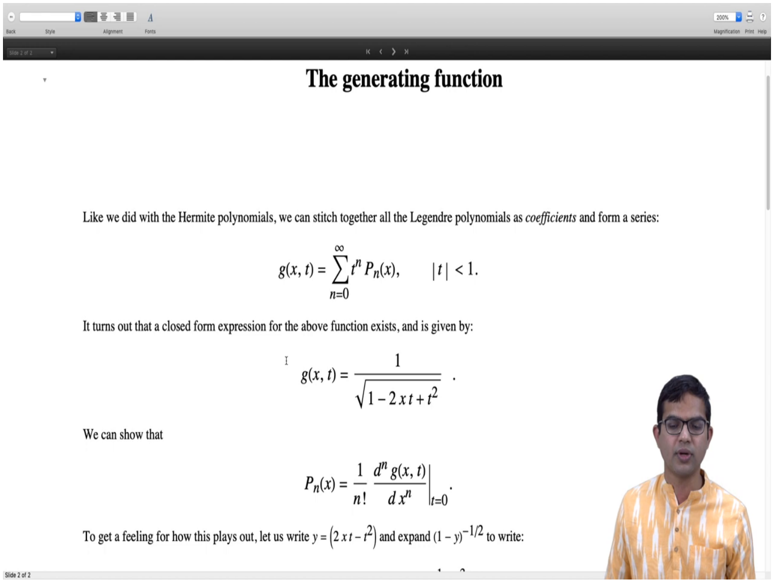

So, what you do is you collect all these Legendre polynomials and then tag them with these powers t to the n, and then form the series right. So, notice this approach is similar to Hermite polynomials, but by convention it is a little bit different from how we did with the Hermite polynomials where we had these factors of 1 by n factorial which was built into this series right.

So, whereas, here it is just simply t to the n times P n of x, and so it so happens that this is a convergent series when mod t is less than 1. So, you have to restrict mod t to be less than 1. And so there is a closed form expression available for this generating function, and it is simply given by 1 over square root of 1 minus 2 x t plus t square. So, this generating function you might have encountered in E and M, but I mean let us see how it operates right.

So, I mean first of all we have to argue that this is indeed the correct generating function. And in order to do that, we have to show that if you take the nth derivative of this function g of x comma t, and then divide by 1 over n factorial right I mean which basically a Taylor series expansion of this function you know about t equal to 0 is supposed to is that is what this expansion is. And these coefficients are functions purely of x. And they have to be given by this formula. If you can show this, then we are done right.

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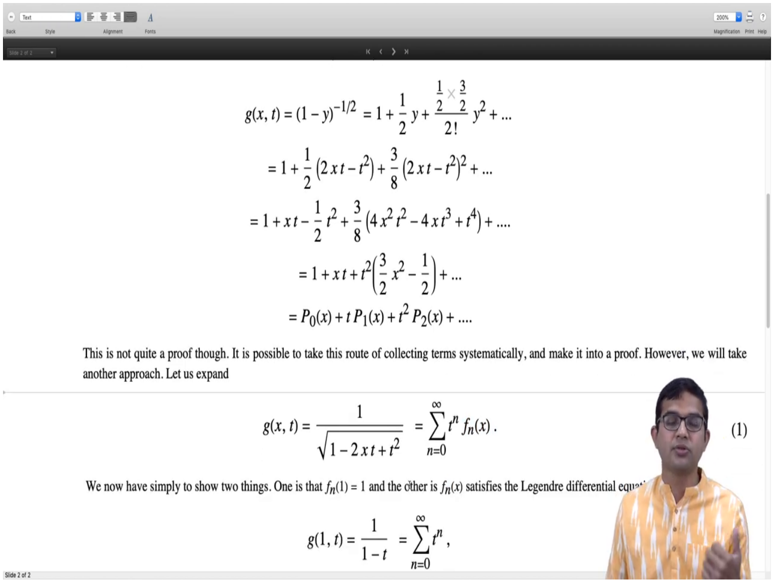

So, to get a feeling for how this plays out, so one way to do this is to just rewrite 2 x minus 2 x t minus t squared as y, and then expand this function 1 over 1 minus y to the whole power minus half 1 over square root of 1 minus y, or 1 over 1 minus y to the minus a half. And then if you expand it, then we have this expansion 1 plus half y plus half into 3 by 2 divided by 2 factorial y squared so on and higher order terms right.

So, if we collect the first three terms right, we already start seeing a pattern. So, 1 plus half times 2 x t minus t squared plus 3 by 8 times 2 x t minus t square the whole square plus so on. And then once again we look at 1, and there is a x t, and then a minus half t squared minus yeah it is just minus half t squared just from here and then we have this 3 by 8 times 4 x squared t squared minus 4 x t cube plus t to the 4.

And then if you just collect all of these, we see that you have 1 plus x t and then t squared times 3 by 2; 3 by 2 x squared minus a half plus higher order terms right, so which we are not writing down.

But basically the point is that if you look at just these first three terms, you know the in the term corresponding to t to the 0, the term corresponding to t to the 1, and the term corresponding to t square the coefficients are exactly the Legendre polynomials. We have already seen that P naught of x is equal to 1, P 1 of x is equal to x, and P 2 of x is equal to 3 by 2 x squared minus a half right.

So, you can continue and then extract more terms. And in fact it is possible to give a full argument along these lines to show that in general the nth term is going to be P n of x, but let us look at another way of showing this. And so it is quite instructive. So, we will sort of sketch it, but I will not give you all the details. So, the details will be part of homework to show that you know g of x t is indeed the generating function by a method which I will sketch now.

So, the idea is that we want to show that you know this function, this closed form expression you know corresponds to this Taylor series involving Legendre polynomials right. So, but first of all we are free to do the Taylor series, and write it in terms of some f n of x. So, what we will have to show is that f n of x is indeed the Legendre polynomials right.

So, it suffices if we show three things, one is first we must show that f n of x is a polynomial, second is that f n of 1 must be equal to 1 for all n. And the third is if you can show that f n of x satisfies the Legendre differential equation basically we are done right. So, we can show that f n of 1 is equal to 1 in a straightforward way.

So, let us do the easy part and leave the difficult part of the exercise for you to check this. I mean indeed all of these are going to be polynomials in x - that is also clear right just from looking at this expansion, you will see that the operations involved are all you know squares and cubes and so on. So, indeed, you are going to get polynomials in x.

And now f n of 1 is equal to 1 follows from the fact that if you put x equal to 1 here. So, you get g of 1 comma t is 1 over 1 minus 2 t plus t squared square root of that which is just 1 over 1 minus t. And then you can of course write this as summation over n going from 0 to infinity t to the n right.

So, we see that all the coefficients are 1. So, indeed immediately this implies that P naught of P naught of x is equal to 1, P 1 of x equal to 1 so on, and in general P n of x is equal to 1. So, in, so at this point, we are calling it just f n. So, f n of 1 is indeed equal to 1 that is immediately evident from this series expansion which you are familiar with.

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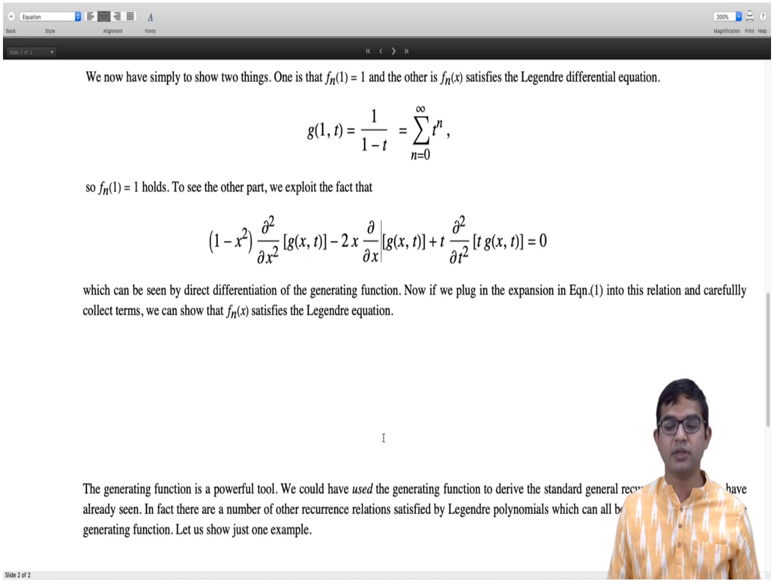

Now, the other part is a little more involved, it requires some algebra, but I will sort of tell you how this comes about right. So, the first step is to show that this function g of x there are two ways of taking partial derivatives of this function g. You can either take a partial derivative with respect to x or with respect to t.

So, it turns out that there is this nice combination of you know partial derivatives, you take a partial derivative with respect to x here, and a second order partial derivative with respect to x. And then you connect it to the second order partial derivative of this function t times g of x comma t right.

So, I am giving you the answer, but the way to see this is to explicitly show this right. So, I mean it is like an exercise in sort of trying to fit it you know in it like a trial and error way perhaps is how one might have discovered this in the first place, but once it is known it is actually very straightforward to show this. All you have to do is take this function and then differentiate it with respect to x partial derivative with respect to x, and a second time, and then combine it with this 1 minus x squared, and then take this you know do a minus 2 x times dou by dou x of g.

And then you can show that I mean in parallel you can also take these derivatives with respect to t, and a second order derivative with respect to t. And then you tag along these you know t is outside and inside in a very well designed way, so that you can show just purely based on you know the information that you have this function g of x comma t, this relation is satisfied.

Now, the next step is to take this expansion for g of x comma t that we have, I mean this is a Taylor series expansion and plug this into this relation. If the function itself satisfies this relation, then the series representing the function must also satisfy this relation. And then we argue that indeed it must satisfy term by term.

So, every coefficient corresponding to t to the n must separately satisfy this relation right and so because it is a unique expansion you have which is valid in some region right, so Taylor series expansion. And therefore, we argue that term by term it is going to hold. And when you do that carefully, you are going to recover exactly the differential equation corresponding to the Legendre equation.

So, I am not going to plug in all the details, and I will not do it here because it is just a lot of algebra, and it consumes a lot of time, but this is going to be homework right. So, you will verify that indeed this relation holds first of all. And secondly by plugging in this expansion, working out dou by dou x with respect to this dou squared by dou x squared with respect to all these functions.

And then you have to work out this quantity, you have to work out this quantity, this quantity. And then collect all of these terms, collect the coefficient corresponding to t to the n, and you know put that individually to 0, then you will see that that is going to be nothing but the Legendre differential equation right.

So, that is basically the argument you know this series of steps which you have outlined is how you show that indeed this is the correct generating function. So, now, what I am going to do is quickly show you how this is a powerful tool. Once you have the generating function for a sequence of polynomials, we saw with Hermite polynomials how it can be useful. And so here again you can use it to derive a number of interesting recurrence relations.

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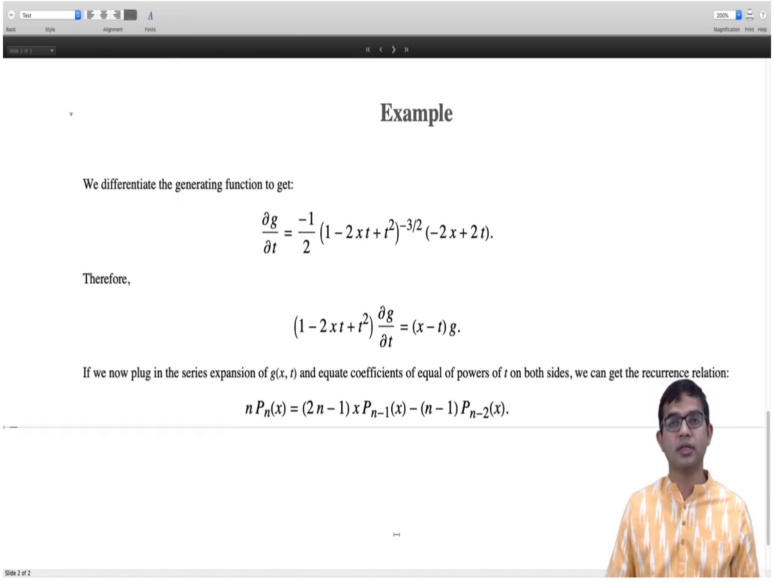

So, let me quickly show you how to derive the so-called three term recurrence relation which actually boils down to just a two term recurrence relation in this case, but let us see how to obtain it from the generating function. So, what you do is you start with g of x comma t and take a partial derivative with respect to t.

So, then you get minus a half times 1 minus 2 x t plus t squared the whole power minus 3 halves times minus 2 x plus 2 t right. So, if you then you pull out this minus 2 which cancels with this 2 and this minus 1. And so basically if you multiply throughout from the left hand side with 1 minus 2 x t plus t squared times dou g by dou t, then you basically you know this factor becomes g again.

So, you get 1 minus 2 x t plus t squared the whole power minus half which is g. And then this you only have x minus t here because you have pulled out this minus 2. So, basically this factor times this dou g by dou t is equal to x minus t times g. Now, we want to yeah. So, once again I am skipping some steps which I would like you to work out the details of.

So, you take this relation which we have shown from the master representation for this generating function. And then you argue that since it is true for the function g of x comma t, it must be also true for the series representation. So, you take the series representation and then carry out dou g by dou t, you are going to get another series, and then multiply by this factor, then you are going to get some series on the left hand side and you get another series on the right hand side.

And then you have to equate you know the corresponding term by term on the left hand side and on the right hand side. And when you do this carefully, if you equate coefficients of equal powers of t on both sides, you would immediately get this relation which is actually nothing but the recurrence relation we have already derived.

Although in the form in which we derived it looked a little different because we had n plus 1 P n plus 1 of x is equal to 2 n plus 1. So, in place of whatever is n here, if you put n plus 1, it is going to be the recurrence relation which we have already derived. So, it will be homework for you to complete this exercise, but there will also be other recurrence relations which I will also spell out and you should be able to derive those as well using this generating function ok. That is all for this lecture.

Thank you.