Mathematical Methods 2

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Orthogonal Polynomials
Lecture - 52
Laguerre polynomials

Ok, so, with this lecture, we begin our study of Laguerre polynomials. So, these are the third kind of polynomials. So, we had these general classes. And so this is going to correspond to the next sequence of polynomials which also fits into this broad umbrella which we came up with, ok.

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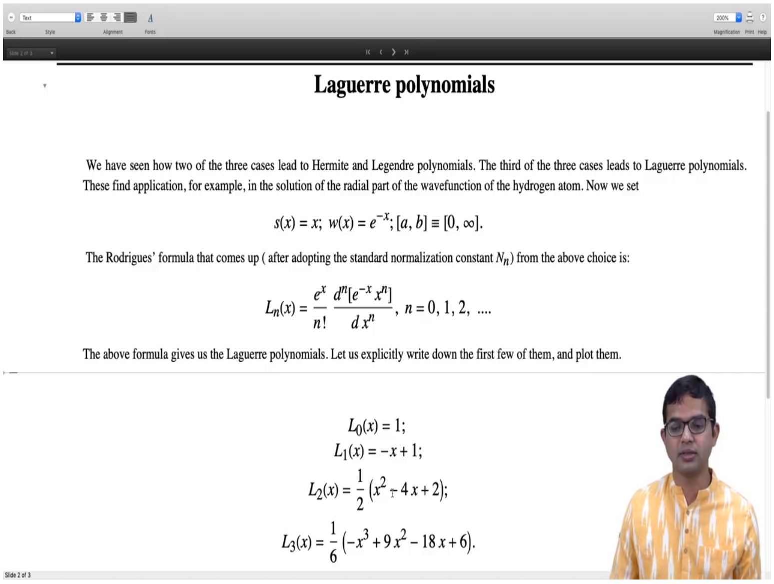

So, the Laguerre polynomials are obtained when we take s of x is equal to x, w of x is equal to e to the minus x. So, this weight function has an important role here. And the limits now a and b correspond to 0 and infinity.

So, we have seen that if a is finite and b is plus infinity, then you can just choose a to be 0 right, so that is what this case corresponds to. And here the weight function has an important role. So, it is taken to be e to the minus x. So, if you do this, the Rodrigues' formula that comes up when you know how to plug in these expressions is the following.

So, L n of x is given by e to the x divided by n factorial where I mean we choose this normalization constant in a suitable way as we will discuss as we go along. And so you have e to the x divided by n factorial times the nth derivative of e to the minus x times x to the n; n equal to 0, 1, 2 and so on.

So, you can check that this formula reproduces for you a bunch of polynomials right. So, you notice that whenever you take a derivative of this product, you're eventually always going to end up with some e to the minus x in every term and that will cancel with this. And you are going to have a polynomial in x, right.

So, let us explicitly write down the first few. So, we see that L naught of x is just 1 because L naught is like not taking any derivative e to x goes with e to the minus x, and then you are just left with x to the 0 divided by 0 factor which is the same as 1. And then L 1 of x you can quickly see is just minus x plus 1, L 2 of x is half times x squared minus 4 x plus 2. And L 3 of x is 1 by 6 times minus x cube plus 9 x squared minus 18 x plus 6. So, all of these are directly obtainable we using the Rodrigues' formula right.

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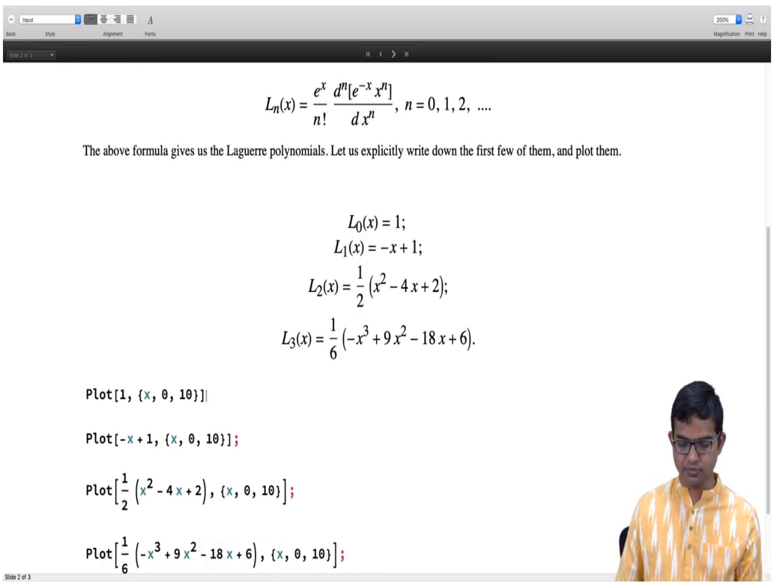

So, for the smallest few it is it's convenient to just do direct differentiation, it is only when you need higher order polynomials that you may want to come up with some other way of obtaining those polynomials ok. So, one observation we make ok, let us plot these function and then I will tell you what this observation which we can also make from the plot ok.

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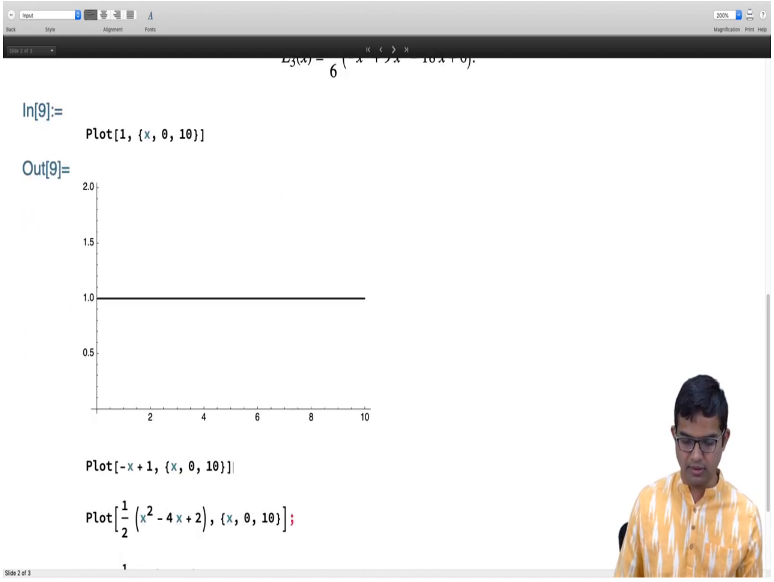

The first function is just constant, nothing much happening there.

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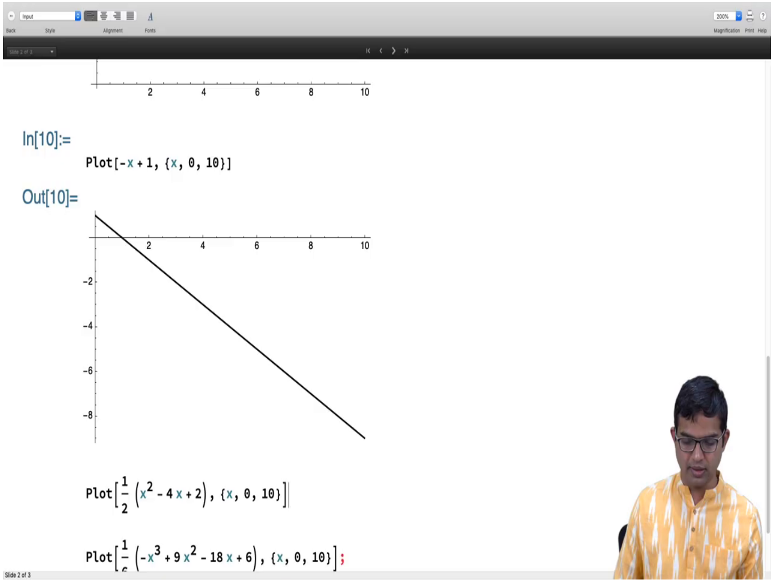

The second one is going to be linear and it is going down as a function of x.

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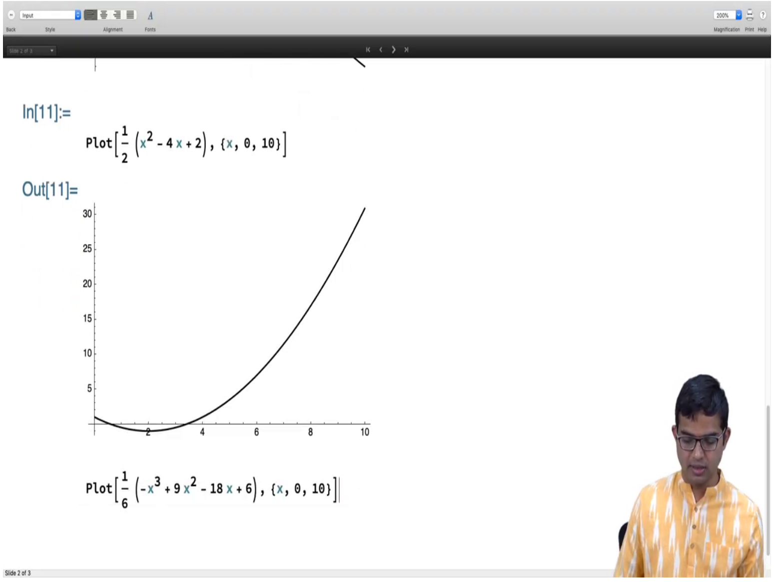

And the third one is a quadratic function, but it is a you know it is it goes down, and then goes back up right. It is negative in some region, but then again it becomes positive.

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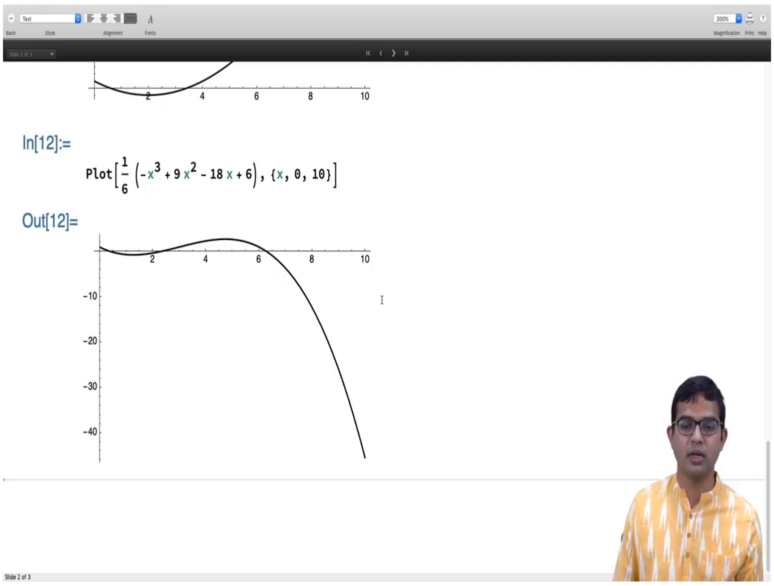

And then we see that the next member of the sequence you know starts going down, then goes back up, and then turns around again and then keeps going down after that. So, there are regions where it is negative, then there is a region where it is positive, and then there is again a region where it becomes negative and large right.

So, clearly these polynomials are you know they do not have to you know become nice and small for large x. They can become quite large positive or negative - the burden of convergence for these inner products and so on will be entirely on this weight function w of x which is exponentially decaying. So, there is no problem, it is definitely going to be able to take care of it.

So, the observation that is useful to make here is that all of these functions start at the same point at x equal to 0, they are 1. And so in fact that is how we set the normalization with the condition that Ln of 0 is equal to 1 right.

So, this is a sequence where it is kind of different from the other two sets we have discussed, because they do not have this definite parity, right. So, it is only the first function which is kind of a trivial one which is just a constant, but every other remember of these sequences it is neither or nor even right as you can check, ok.

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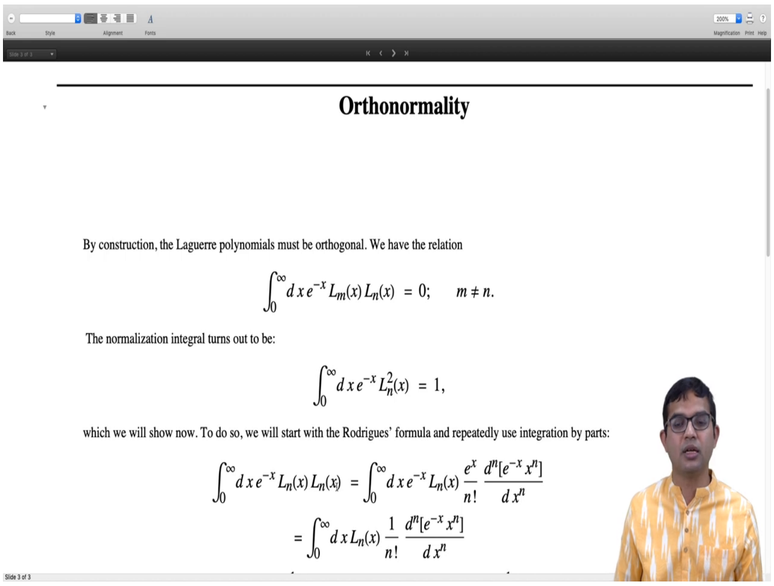

So, let us move on. You know the first order important property of Laguerre polynomials is the orthonormality of after all these are orthogonal polynomials. So, let us write down this orthogonality property.

And so the orthogonality property by construction is that you know you take the weight function and multiply any two polynomials, and integrate inside the limits specific to those the set of polynomials. In this case, it is from 0 to infinity d x e to the minus x L m of x times

Ln of x if this integral must be 0, whenever m is not equal to n. And the normalization integral, it turns out to be very straightforward to write down and we will show how this comes about. So, if you integrate from 0 to infinity d x e to the minus x L n squared of x is going to be 1, it is just as simple as that.

Now, let us show this. So, to do this, we will start with the Rodrigues' formula and use integration by parts repeatedly in a manner very similar to how we did with the earlier two examples. So, we say 0 to infinity d x e to the minus x L n of x squared the same as L n of x times Ln of x. And one of these L n of x we just leave it as it is, but the other L n of x we replace it by the Rodrigues' formula. So, we have e to the x divided by n factorial the nth times nth derivative of e to the minus x times x to the n.

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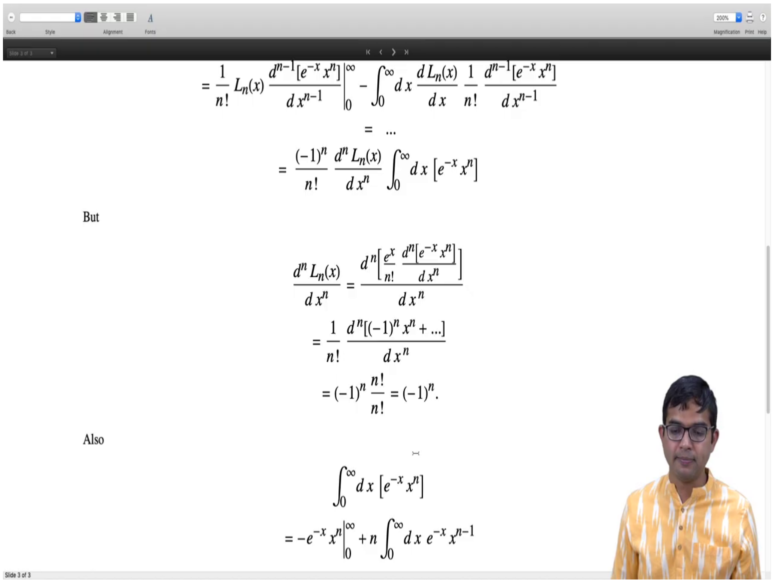

And so then we observe that this is you know this e to the minus x will cancel with this e to the x. And so we just left with this integral 0 to infinity d x L n of x you know this one by n factorial also we can pull out. And so basically we have some u d v alright. So, there is a function times a derivative. So, whenever you have this, immediately it suggests that we should try integration by parts and indeed that works out very well here.

So, we write down the integral u d v is the same as u v evaluated at the two ends. So, L n of x times the n minus 1-th derivative of this stuff e to the minus x times x to the n 0 to infinity minus integral 0 to infinity d x, then you have to take a derivative d u, and then v 1 over so here it is just 1 over n factorial the n minus 1-th derivative of this function, right.

So, now we argue that this quantity for the boundary term is actually going to go to 0 at both ends separately at plus infinity and at minus at 0. At 0, it goes to 0 because you have x to the n right.

So, I mean I am assuming that n is at least 1 right. So, you can check explicitly finally, that this result also holds with n equal 0 right, I mean, but let us say that n is greater than or equal 1 at this point. So, when n is greater than or equal to 1, indeed the value of this integral at plus infinity of course e to the minus x will kill it. And at x equal to 0 you have this other term which is going to kill it, right.

So, let us say that you have a sufficiently high power of x, I mean maybe even for x equal to n equal one also you can explicitly check it n equal to 0, n equal 1, n equal to 2; the first few terms you can explicitly check this quantity. That any case for higher powers will argue that you know you have a e to the minus x times x to the n. And so you are taking only the n minus 1-th derivative of this. So, every term is going to have at least 1 x sitting there right.

So, if it is I mean yeah, so it also will hold for n equal 1. So, if you have x times e to the minus x and then you take a derivative well I mean you do not take a derivative. So, it just e to the minus x times x. So, n minus 1 is just 0. So, there is no derivative involved. So, indeed x equal to 0 is going to 0, e to the minus x also is going to go to 0 x plus infinity.

So, this is going to be 0 at both ends. So, then you are left with just this minus 0 to infinity d x the derivative of this function times one over n factorial n minus 1th derivative of this stuff. And then we connect the wheel, again I mean it is the same method applied again and again until we reach a point where there are no more derivatives; there is no more d by d x left right.

But I mean the key point is that every time this bounded term is going to keep on going to go to 0 at both ends because of the presence of e to the minus x and x, you know at both limits there are factors which will ensure that it is just 0. So, the boundary terms do not leave. And then you will keep oscillating between minus 1 and plus 1.

So, you will have a factor of minus 1 to the n when you keep doing this n times. So that is going to be come up points, so the next time you going to get the d squared of L n of x, then d cube and so on and when you do this n times you are going to get the nth derivative of Ln of x which you can pull out it does not have to live inside the integral the reason is L n of x is a polynomial of degree n. So, the nth derivative of a degree n polynomial is a constant. So, the constant will come out.

And there are no more derivatives left here. So, this function will remain as it is. So, all we have to do is work out this nth derivative and work out this integral and then we are done, right. So, in order to do this let us start with this nth derivative of Ln of x yeah.

So, this will go back to the Rodrigues' formula. So, the nth derivative of this is the same as nth derivative of this whole stuff which is in turn which can be written as 1 over n factorial nth derivative of minus 1 to the n times x to the n so on right.

So, the idea here is that if you are going to take the nth derivative with respect to x, there is going to be a bunch of terms here. And well I mean you are going to I am talking about this nth derivative. So, you are going to take a nth derivative of a polynomial. Now, the only term which will remain is the highest order term, everybody else is going to just go to 0.

And what is the highest order term that is going to come when you know in each step you take a derivative only with respect to e to the minus x. You have to leave this entire x to the n s itself because if you take a derivative with respect to x to x to the any one once, then the order of the term is going to be lower than n. And then when you take this, this nth derivative outside it is going to not contribute.

So, the only term we are interested in is the first one, the highest order term which corresponds to just minus 1 to the n time e to the minus x, but this e to the minus x will go away with e to the x, and you are just left with x to the n.

And then we have pulled out this factor 1 over n factorial outside. And so it is very straightforward to see that the answer now is going to be just you know taking a derivative of x to the n times, and that is going to give you n times, n minus 1 times, n minus to all the way down to 1s, so that is n factorial. So, this n factorial will cancel with this n factorial in the denominator, and you are left with which minus 1 to there right. So, straightforward to see that the nth derivative of Ln of x is just minus 1 to the power n.

Now, I mean this is also something that we saw explicitly in the first few terms that we wrote down. So, the nth derivative is basically it is like the constant term. So, you have a 1 here, then you have another 1 here upon here, and so on. So, well, I mean you have to be; you have to be careful, but basically so it is just. So, let us see.

So, if you take the nth derivative of, so the first derivative, and so if I mean it is the first term. So, in this case, it is just 1, then you have a minus 1, then you have 2 times x, it is going to be a plus 1. So, you get a minus 3 x squared times to x, it is going to give you a minus. So, you see indeed going to be exactly like what we have seen here minus 1 to the n ok.

So, let us work out the other integral which is a familiar integral 0 to infinity d x times e to the minus x to the n which is you know again we will do this by parts. So, it is u d v. So, it is you write the first term as so; I mean u here is e to the x to the n and d v is e to the minus x.

So, you write it as minus e to the minus x time x to n 0 to infinity plus so then we have to take a derivative of this function x to the n. So, you get an n x to the n minus 1. And there is this minus which combines with the minus from this integral, and so you get a plus n time 0 to infinity d x e to the minus x to the n minus, and then we keep repeating it. And first of all argue that this boundary term is 0, because it is 0 at plus infinity and it 0 at 0 separately because of the factors x and e to the minus x.

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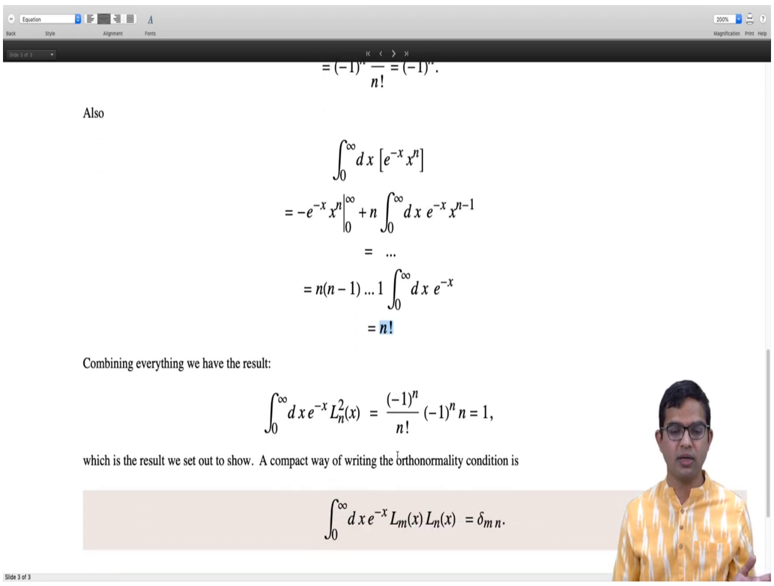

So, all these boundary terms will go to 0. And then you keep repeating this in a recursive way, and then you get n times n minus 1, times n minus 2, all the way down to 1. And you will be left with just this integral 0 to infinity d x e to the minus x which you can evaluate, and it is going to be just 1 right. So, this is just n factorial.

So, basically we have managed to show that this integral, the normalization integral is minus 1 to the n divided by n factoria1, so that is this guy, and then we have a minus 1 to the n which comes from here. And then we have an n factorial which comes from here. And so if you take the product of all these it is just 1, right.

So, ultimately you know the normalization integral is just 1 for every index involved, and so the compact way to write down orthonormality condition is just to say integral 0 to infinity d x e to the minus x L m of x time L n of x is equal to Kronecker delta m n ok.

So, that is all for this lecture.

Thank you.