Mathematical Methods 2
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Bessel Functions
Lecture - 60
Bessel functions: Orthogonality

Ok. So, we have looked at a bunch of properties of Bessel Functions, which you know or some of these properties we have seen are like the orthogonal polynomials we discussed earlier; Bessel functions are not polynomials. However, you know there is this lot of analogy one can draw and there are ways in which we can connect the two kinds of functions.

And there is in this lecture we will look at you know one more such feature namely orthogonality right. So, there is a sense in which Bessel functions are also orthogonal with respect to each other with respect to weight function and we will look at some of those details in this lecture, ok.

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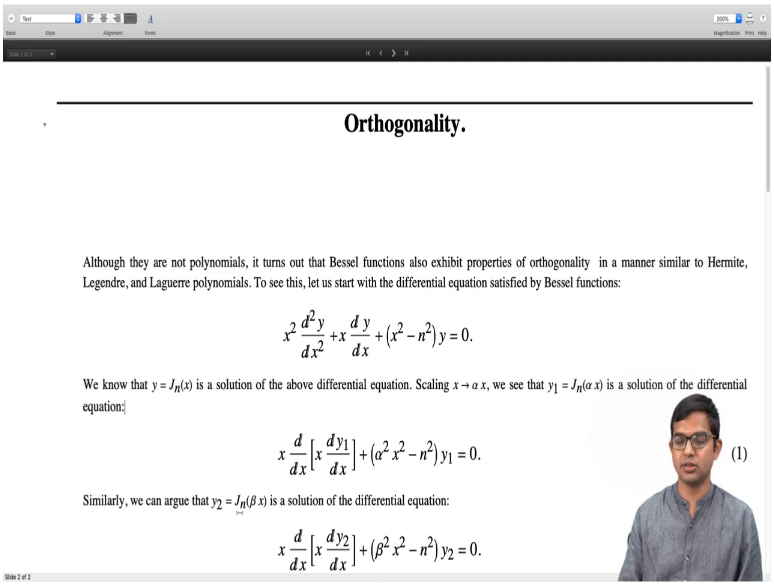

So, in order to you know workout the orthogonality properties of Bessel functions we will start with the differential equation right that are satisfied by Bessel functions that we have seen how x square d squared y by d x squared plus x d y by d x plus x squared minus n squared times y equal to 0 is the differential equation corresponding to which we have the Bessel functions of order n; J n of x you know is a solution of the above differential equation.

And so, it is convenient for the discussion I had to scale this variable x by a factor of alpha. And so, if you, you know, take x and rewrite it as alpha times x we see that you know y 1 it is going to be a solution y 1 is equal to J n of alpha times x is a solution of this differential equation, right.

So, in place of x, I put alpha x. So, x square and you know this alpha squared and alpha squared in numerator and denominator will cancel and then and the second term again I have x and d x in the denominator. So, alpha appears in both numerator and denominator. So, it does not really change anything.

It's only you know when you have x squared minus n squared that is going to become alpha squared x squared minus n squared. So, in place of y I have put y 1; y 1 simply means that the argument is alpha times x. And now it is convenient to rewrite these first two terms in this form, right.

So, you can pull out this factor of x outside and so, we have x d squared y by d x squared plus x plus d y by d x. So, we can rewrite this as d by d x of x times d y 1 by d x, right. So, if you expand this you will see that you indeed get back these two terms. So, if I differentiate with respect to x then I will get just this first term x squared times d squared by 1 by d x squared, but if I differentiate with respect to; with respect to x then I will just get the second term right.

So, that is just the convenient way of rewriting the sum of these two terms. And then this is the term where you get alpha squared x squared minus n squared and then we have a y 1 sitting outside yeah, outside this factor and that is equal to 0.

Now, similarly we can argue that if we had scaled this by some other factor beta right and if we you know consider y 2 is equal to J n of beta x its going to be a solution of this differential equation; x d by d; x times d by d x of x time d y 2 by d x plus beta squared x squared minus n squared times y 2 is equal 0, right.

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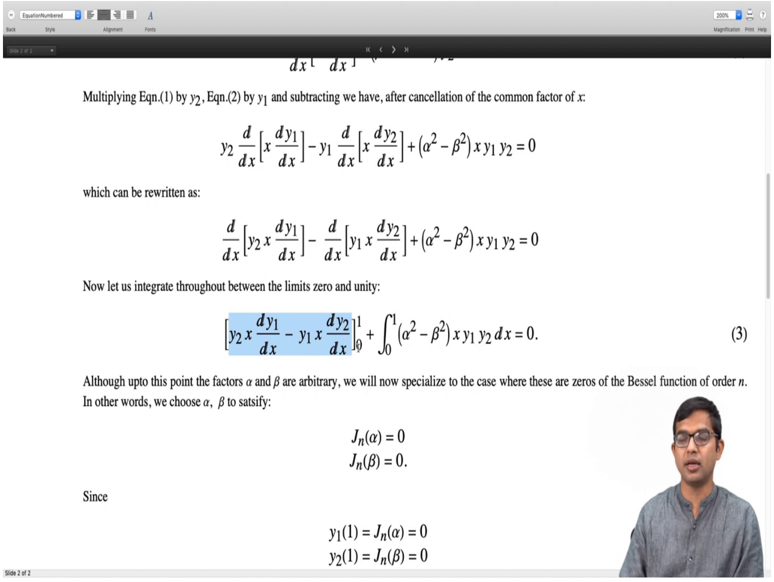

So, now we will use these two equations, equations 1 and 2 and then try and eliminate some stuff. So, convenient to multiply equation 1 with y 2, equation 2 with y 1 and then subtract right. So, we get y 2 times and then there is an overall factor of x which will come out and which you can take it to 0.

So, the reason that is possible is because you know you have n squared times y 1 into y 2 and then you have an n squared times y 2 into y 1 these two will cancel, and then every other term. So, this term has an x squared and this term has an x. So, you can cancel one x throughout. And so, basically you have y 2 times d by d x of x times d y 1 by d x that is the term from here from equation 1.

And then there is a similar term which comes from here which is you know the x goes away then you have a y 1 times d by d x of x times d y 2 by d x, right. So, these two terms you know go together. And again you have alpha squared minus beta squared times x square will become x because of the cancellation of this factor of x times x times y 1 y 2, right. So, there is a y 1 y 2 here and there is a y 2 y 1 here. So, which is can write this whole thing as this equal to 0.

Now, which in turn can be rewritten as a you know the first two terms can be written as a total derivative right. So, you see that there is this y 2 you can actually push this y 2 into this under this derivative because you also you can push; if you also push y 1 into this. Because basically you get an extra term here and an extra term here which cancel each other exactly, right.

So, if you look at this for example, you see that you know the derivative of this product is really can be thought of as d y 2 by d x times x times d y 1 by d x minus you know the first time when you take a derivative with respect to y 1 is going to become d y 1 by d x times x times d y 2 by d x.

So, the extra term that is added you know that comes out here will get subtracted with the extra term here. So, basically you will get back only this term which will survive here. But, it is convenient to write it like this because clearly you see that we can even think of this as d by d x of this stuff minus this stuff right which will go to be convenient as we integrate next, right.

So, this term we have left it as it is. So, the next step of course is to integrate throughout in the limits between the limits 0 and 1, right. So, the first two terms are really a total derivative. So, you can simply write it as y 2 x d y 1 by d x minus y 1 x d y 2 by d x and with limits going from 0 to 1. You will see that the limits 0 and 1 are convenient and you will have certain nice cancellation, which happens.

And then you have this integral 0 to 1 alpha squared minus beta squared you know this whole factor times x times y 1 times y 2 d x is equal to 0. And now up to this point we have left alpha and beta to be arbitrary, but now we will specialize to the case when these alphas and betas are actually 0s of the Bessel function of order n, right. So, you will see in a moment that when you do this more simplifications are possible and then we will be immediately led to the orthogonality condition.

Now, if you choose alpha and beta to be roots of the Bessel function then what it means is you choose alpha and beta such as J n l of J n of alpha is equal to 0 and J n of beta is equal to 0, right. So therefore, you know here we have to take these limits right. So, you see that this whole stuff definitely has to go to 0 at x equal to 0 because you have this factor of x here and a factor of x here. So, the lower limit for sure is 0. It is only the upper limit that we have to try and work out.

But, here if you choose alpha and beta to be you know the roots of the Bessel function immediately actually gives us also the upper limits individually each of these terms go to 0.

The reason is you have a y 2 of 1 and a y 1 of 1 sitting here, but y 2 of 1 is nothing but J n of beta x where x equal to 1. So, that is J n of beta and again y 1 of 1 is the same as J n of alpha times x where x equal to 1. So, it's J n of alpha.

Both of these by the choice that we have made that these alphas and betas are I will find beta in this case, but we can specialize to you know alpha and beta can be any of the infinitely many 0 so that Bessel function of order n has and that is it. So, therefore, immediately we see that both these terms are 0. And so, we are left with this equation.

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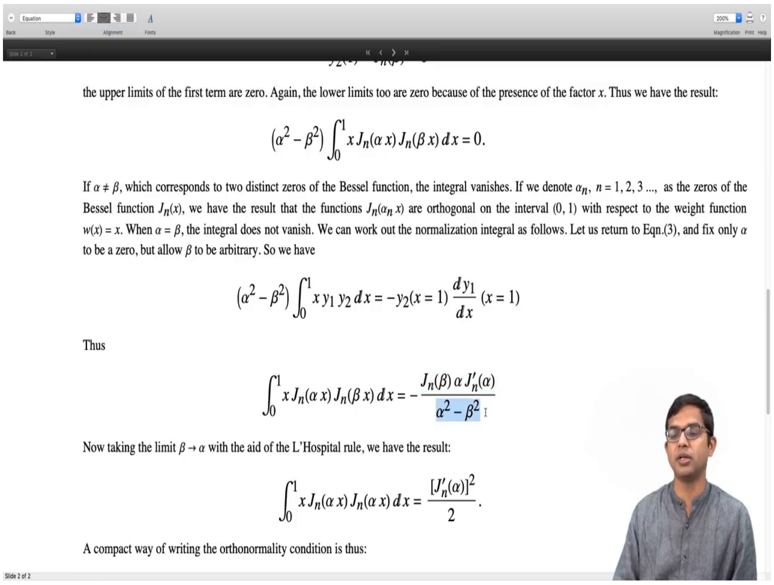

If you have chosen alpha and beta to be you know roots of the Bessel function then we have this result alpha squared minus beta square times integral from 0 to 1 x times J n of alpha times x times J n of beta times x the whole thing multiplied by d x this integral is 0, right. So, if alpha is not equal to beta, the only way that can happen is if this factor does not take it to 0 then this integral must be 0, right.

So, that is really the core of the orthogonality condition right. So, if you have, if we denote this alpha n, n going from 1, 2, 3 etcetera as the 0s of the Bessel function J n of x we have the result that the functions J n of alpha n x are orthogonal. We might call it a result; the result that the functions J n of alpha n are x are orthogonal on the interval 0 to 1.

So, the interval is important with respect to the weight function w of x equal to x. So, this is very similar to the kind of you know integrals we wrote down as the orthogonality conditions when we work with orthogonal polynomials, right. So, there was a weight function then there was an interval and so, Bessel function is true. Although they are not polynomials they are in fact, we define them in terms of a series, there you need an infinite number of coefficients to define a Bessel function.

But really they have some properties which make them similar to these orthogonal polynomials. So, they have this orthogonality between Bessel functions of the same order. If you take two Bessel functions of the same order and you know tag them along in this manner with 0s coming from two different 0s, two distinct 0s and then you will get this orthogonality condition.

Now, if alpha is equal to beta then of course, this integral need not be 0 and in fact, it is not 0, right. We can work this out, that is the normalization integral right. So, in order to do this actually we can go back to equation 3, right. So, we want to work out this normalization integral.

So, you see if you look at this equation this is completely general there is no condition or what alpha or beta should be up to this point. So, if you go to this point in this equation and say that we choose only alpha to be a 0 of the Bessel function, we do not choose beta to be a 0 of the Bessel function, it can be some arbitrary quantity then we have this condition right.

So, alphas alpha squared minus betas. So, this part remains as it is. So we, so this stuff remains as it is, it is only here. So, of course, when you put x equal to 0 both of these terms individually are 0 and y 1 is also going to go to 0 because we have chosen alpha to be 0, but beta is allowed to be arbitrary right. So that means, we get this term. So, this term remains, this term remains only y 1 is 0 at 1, y 2 of 1 is not known we will just leave it as it is.

So and x equal to 1, so we get y 2 upon times d y 1 by d x, but d y 1 by d x is 1, we know how to write it. So, we have a minus y 2 of x equal to 1 times d y 1 by d x at x equal to 1, but y 2 of x equal to 1 is actually nothing but J n of beta x, right. So, y 2 is J n of beta x and if you put x equal to 1, So, you get J n of beta here and then the derivative of y 1 with respect to x is nothing but the derivative of J n of alpha x.

So, that gives you J n prime n of alpha x times alpha, but x equal to 1 is also there. So, the numerator is just this and then of course, this factor alpha squared minus beta squared I can push it to the other side. So, I have this condition you know for if beta is an arbitrary real number and alpha is our 0 of this Bessel for a function order n then we have this result.

So, now we will take the limit of beta going to alpha using the L'Hospital rule right. So, then we have. So, we have to just take the numerator and differentiate with respect to alpha with respect to beta, take the denominator differentiate with respect to beta and then take the limit beta going to alpha.

So, we immediately see that this will give us just J n prime of beta and then we have to go to beta equal to alpha. So, you get J n prime of alpha the whole squared and then we have this alpha sitting here denominator will give you a minus 2 beta evaluated at alpha that is minus 2 alpha. So, minus cancels with this overall minus sign.

Therefore, we immediately get this result that this normalization integral is actually nothing but J n prime of alpha the whole squared divided by 2.

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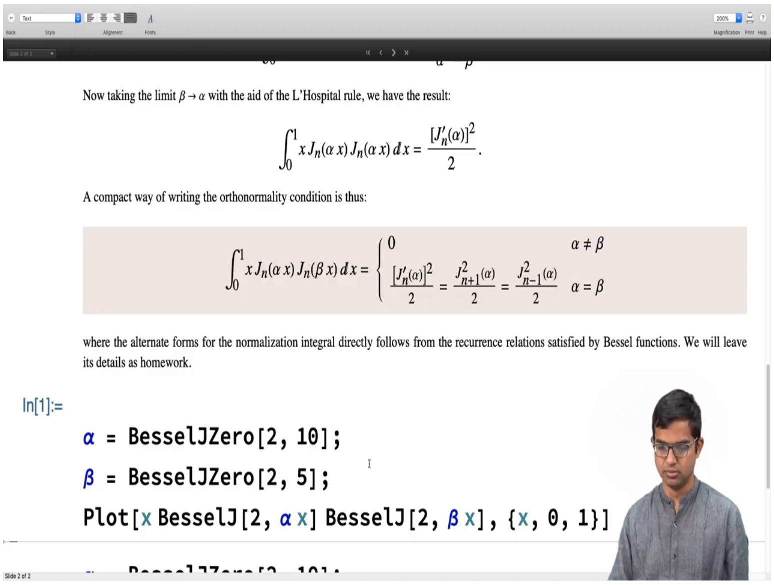

So, a compact way of writing this orthogonal normality condition is like here. So, this integral 0 to 1 x times J n of alpha x; x is the weight function and the interval is from 0 to 1 J n of alpha x times J n of beta x d x is equal to 0, if alpha is not equal beta and it is equal to J n prime of alpha the whole squared divided by 2 if alpha equal to beta. And it turns out that this J n prime of alpha squared can be written as just J n plus 1 squared of alpha or equivalently it can also be written as J n minus 1 squared of alpha, right.

So, we will allow this to be homework for you to explicitly check this. It follows directly from the recurrence relations that we showed for Bessel functions. And so, this is a you know compact way of writing out the whole orthonormality conditions. So, let us quickly look at a few plots of this right.

So, I mean what I have done is I have defined this alpha as Bessel. So, this is I have taken n this n here to be 2 right and I have taken let us say that I am considering you know, I have this freedom of without loss of generality I can take any you know alpha to be any of the zeros. Let me take it to be the 10th zero right. So, this is some syntax for this. So, if you want to find out the 10th zero of this Bessel function of order 2. So, this is how it is.

And then let me compare it with this 5th zero right, so, alpha is this and beta is this and then I am going to just plot this function x times Bessel function of you know order 2 comma alpha x. So, this is the argument which shows us you know whatever is here. Alpha x is here and alpha is defined like here and then times again a Bessel function of order 2 you know of beta x and I am plotting it between 0 and 1.

Let me do this. Hopefully, it will get them sooner than later, it is running. It takes some time for this to run because I guess it has to compute this numerically yeah. So, there you go.

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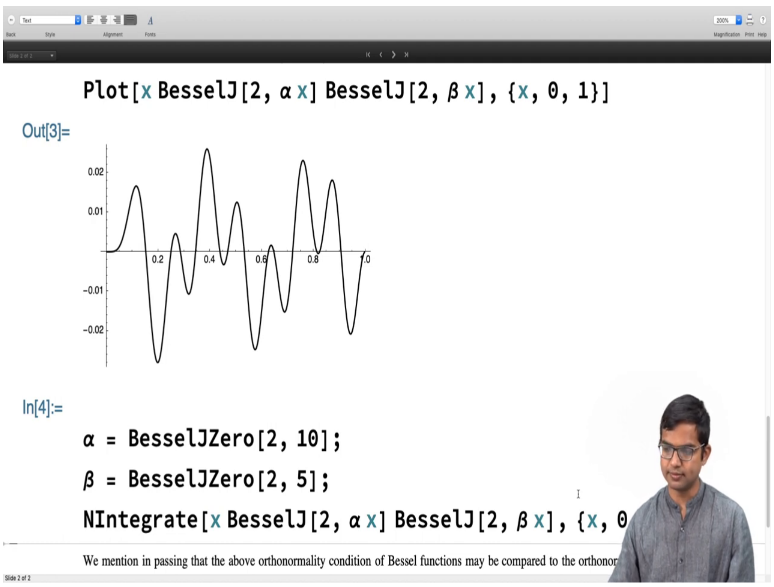

So, the idea is that you see that this function is roughly spending almost as much time above as it does below, right. So, and it is going to go if you find the area under this curve it is going to be 0, right. We have already seen that is the orthogonality condition, right.

So, I could check this explicitly by getting Mathematica to do this integral. So, let me see if I have okay; I have to set this to be 5. If I do this and then it is basically the same code, but now I am going to do an integration let me see what value it will give me. It is going to give me, yeah. So, because there is a numerical integration involved it takes some time, but it also gives me a fantastically small number, it is supposed to go to 0 yeah.

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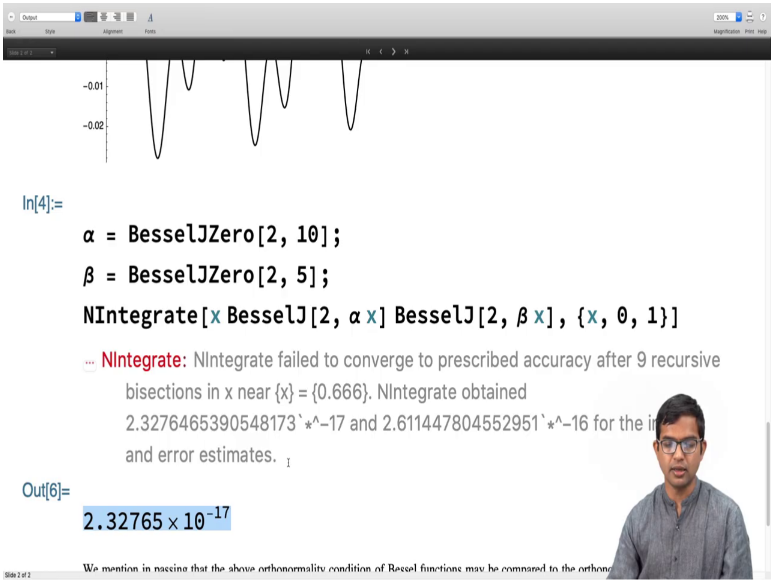

You see there are difficulties because how small this number is basically the point is that it is going to go to 0, right.

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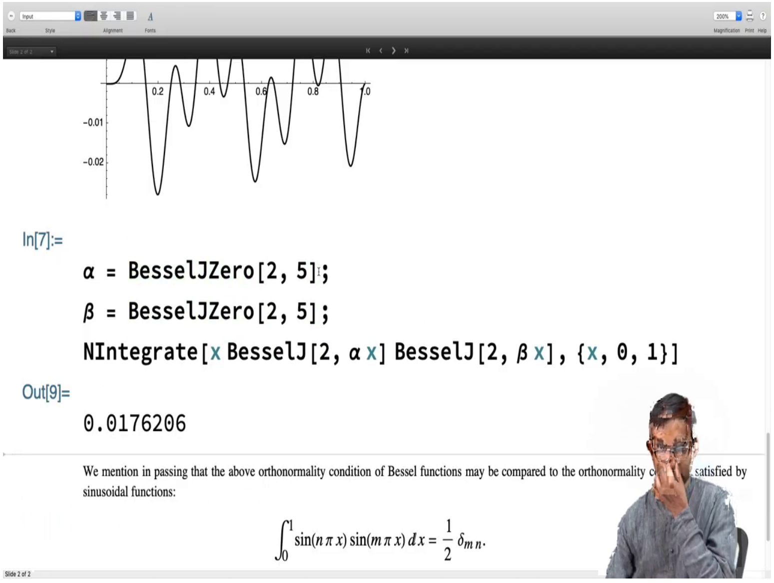

In comparison, let us see what happens if I choose alpha and beta to be the same. If I take alpha and beta to be both 5 and if I redo the same calculation I see I get a value which is not 0.

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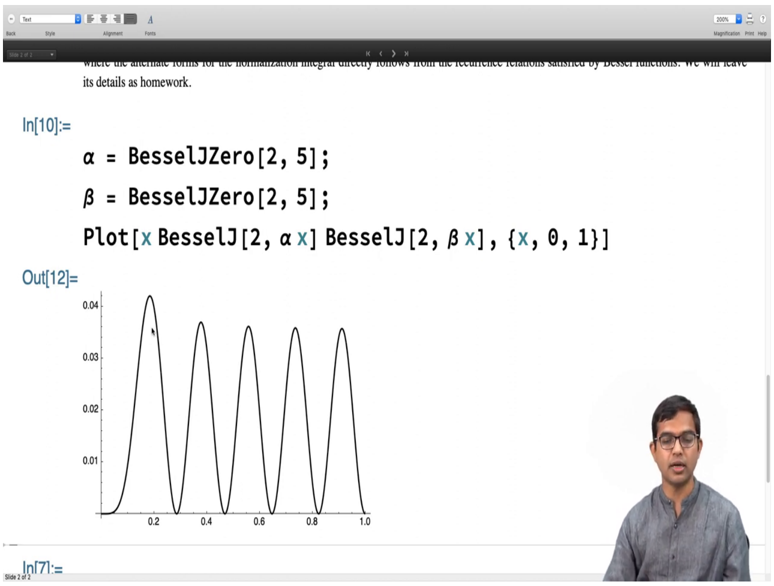

And in fact, I could do that here as well. If I do I could get a 5 and then if I get it to plot, so, this plotting takes some time because I think it has to be numerical - there you go. You see that all these values are above 0 and not very large values. And there is a sort of falling tendency, but clearly you can see that if you integrate this curve it is not going to go to 0. It has to be a positive number, right.

You can play with this and let us you know check for yourself, that you know not only it has it not go to non-zero value, but we know that the value of this integral is exactly this, right. So, if you have access to this you might play some games and convince yourself that indeed it goes to J n plus 1 squared of alpha divided by 2, right.

So, one final comment I want to make is that you know this orthonormality condition is somewhat like the condition that we are familiar with. So, these Bessel functions are also somewhat like sinusoidal functions right. So, sine's; what do sine's do? They oscillate and then they keep repeating.

And Bessel functions also do something like that except that there is no simple way of figuring out exactly where the 0s will appear, that is one thing. And the other is there is a sort of damping associated with a Bessel function; you know the height of each crest is going to be slightly lower than the previous one. It keeps decaying as a function of x right. In a sinusoidal curve you do not get any decay right. It is going to be exactly the same point to which it will return every single time.

And you see here from 0 to 1 again you can think of this n times pi as the 0s of this sinusoidal function, m pi is you know is another 0 of this function. If n and n pi and m pi correspond to different 0s, I mean here the weight function is just 1, right. So, this is a familiar relation and then we get a half. You can check this if you will get a half if n pi is equal to m pi, otherwise you will get a 0, right.

So, this is a comment to conclude our discussion of Bessel functions and then we will move on to next topic after this.

Thank you.