Mathematical Methods 2

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Module - 07
Partial Differential Equations
Lecture - 62
Canonical Forms for Hyperbolic PDEs

Ok. So, we have seen how second order differential equations of a very general type can be classified into three types; hyperbolic, parabolic, and elliptic PDEs. So, in this lecture, we will see how we will concentrate on the hyperbolic PDEs and we will show in detail how one can make a transformation to bring it to the canonical form, ok.

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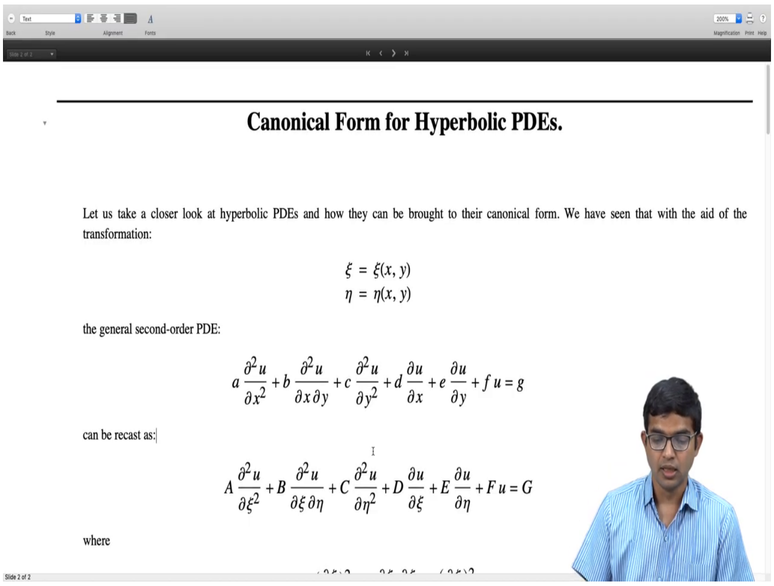

So, we saw how you know there is this transformation zeta which is some function of x comma y, and eta function of x comma y, which can take your general PDE of this kind and rewrite it as a different PDE, but of the same type, right. So, in other words, specifically d squared minus for 4 ac here is the sin of b, small b squared minus 4 ac is going to be the same as the sin of capital B squared minus 4 times capital A times capital C, right.

So, in this transformation, where zeta and eta are independent new coordinates you can think of them as. And so, the reason why we want to do this is we can take this PDE which in general is in a more complicated form and bring it into a simpler form, right. And it is what is called a canonical form. And once you have the canonical form, you know we will see how sometimes it is possible like immediately just simply write down the solution. So, that is the advantage of getting the canonical form.

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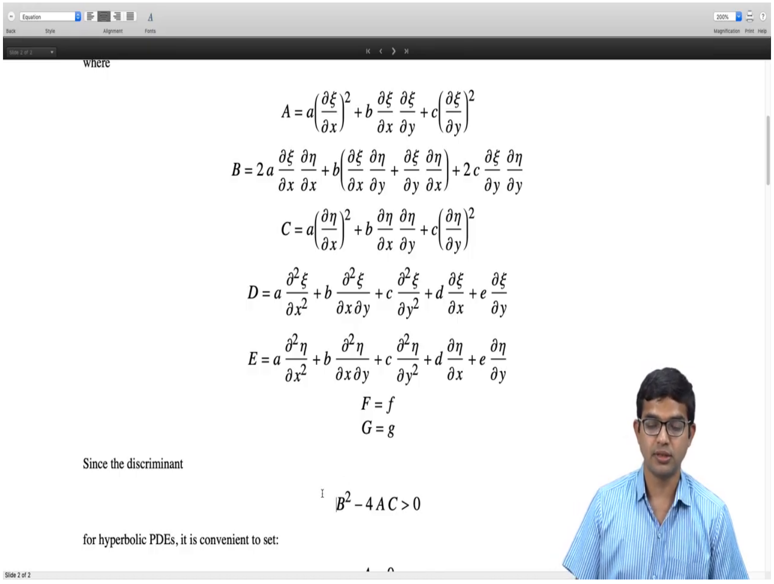

So, we also worked out in some detail how these coefficients are related. A is, capital A is related to you know small a, small b, small c, and all these partial derivatives, right. So, this comes from just some careful bookkeeping, right. So, we take this transformation. And you know we have seen how it is possible to go from one set of variables to the other set of variables and make use of the chain rule and work inside.

Capital A is related to the small a, b, c, in terms of this and these partial derivatives in this manner. Likewise, B can be written like in here. C comes about from here, C somewhat similar to the expression for A. Then, we have the this expression for D and this expression for E, F and G, really remain unchanged, right. So, these F and the G are the last two terms.

So, if you have not already done this, you should go back and look at the previous lecture and make sure that all these equations are correct and you can directly check them for yourselves, right.

So, we have seen how you know the discriminant of either of these differential equations. So, all the information about the type of the differential equation is contained in these first 3 terms a, b, and c; small a, b, and c, or capital A, B and C, does not matter. We just look at B squared minus 4 AC. And in this lecture, we are focusing on the hyperbolic type. So, B squared minus 4 AC is greater than 0 for hyperbolic PDEs.

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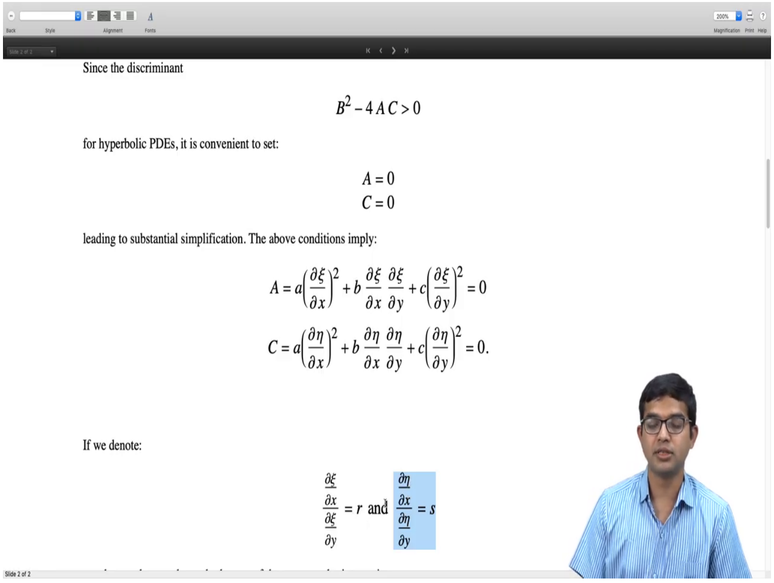

And when this is the case, it is convenient to choose our transformation such that both A and C are 0, right. So, after all, why do we want to make a transformation? We want to make our transformation, so that the resulting PDE is simpler to work with, right. So, let us set A, capital A and capital C to be 0, and so, this immediately will actually lead to substantial simplification.

So, what it does is, so we have this expression for capital A and we have this expression for capital C. So, we; so, this if you are going to put both of them to 0, it means that we have to choose our transformations in such a way that these expressions are both equal to 0, right. So, this I mean there is a way to re you rewrite this in terms of a quadratic equation, right. So, in order to do that it is convenient to define these two quantities.

So, the ratio of these derivatives is dou z by dou x divided by dou z by dou y, if I am going to call it r, and dou eta by dou x divided by dou eta by dou y, if I call this s.

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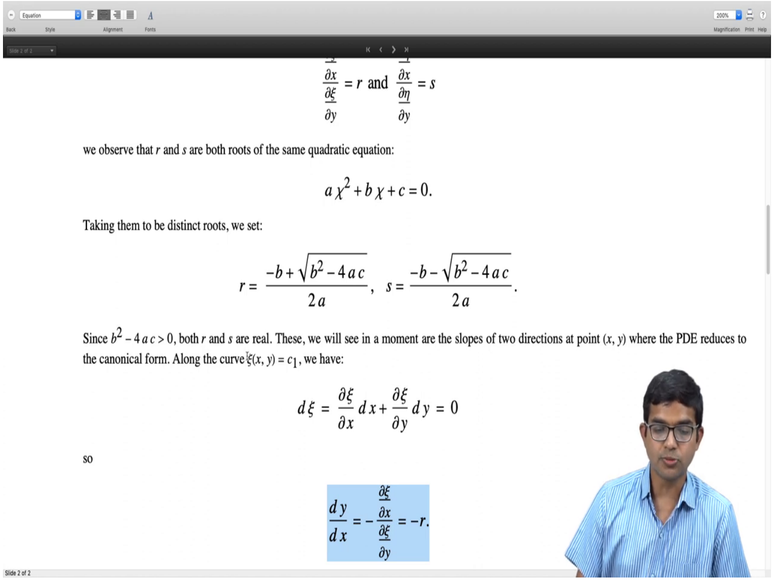

Then, we observe that in fact, both of these, I mean both of these are really quadratic equations. You can divide throughout with dou zeta by dou y the whole squared and likewise you can take this and divide throughout by dou eta by dou y squared.

So, then you get a quadratic equation in dou zeta by dou x divided by dou zeta by dou y or equivalently in terms of dou eta by dou x divided by dou eta by dou y. That is in terms of where r or s are the you know roots, basically they are roots of the same quadratic equation, which I am writing it as a chi squared plus b chi plus c.

So, if I take these two to be distinct roots, right. There are two roots for this quadratic equation and I take r to be one of the roots and s to be the other root, then I can write this as r is equal to minus b plus square root of b squared minus 4 ac divided by 2 a and s is equal to minus b minus square root of the discriminant b squared minus 4 ac the whole thing divided by 2 a. So, this is a parabola, this is a hyperbolic differential equation PDE.

Therefore, we know that b squared minus 4 AC is greater than 0. So, both r and s are real numbers, right. So, these will turn out to be slopes of some very special directions, right. So, you should remember that we are at some point x comma y, and we are making a transformation to some other two coordinates zeta and, zeta and eta.

Now, we can think of these two special directions at the point x, x comma y, along which this curve zeta of x comma y remains a constant, right. Zeta of x comma y at that point itself takes some value.

So, if you look at a direction along which this zeta of x comma y is the same, then we can write d zeta is equal to dou zeta by dou x times dx plus dou zeta by dou y is equal to times dy is equal to 0, right. Because it is a constant along this curve. But what is this curve? That curve is really nothing, but it is given by this differential equation d v by dx is equal to minus this ratio dou zeta by dou x divided by dou zeta by dou y.

But, we have already worked out this ratio - it is minus r. So, if we can solve for this, we get this curve you know corresponding to one of these directions, special directions at the point x comma y.

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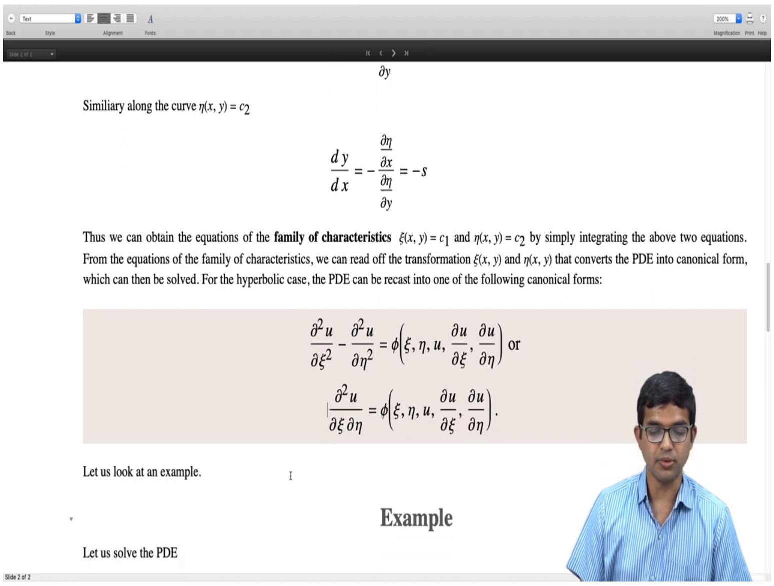

And likewise, if we look at the other curve eta of x comma y is equal to c 2, then that is going to correspond to d v dy by dx is equal to minus s, right which is the other root for this quadratic equation, right. I mean, we choose these two distinct roots, otherwise we will basically get the same direction, but there are actually two different directions corresponding to x comma y, you know along which you know these curves remain invariant.

And so, these are called characteristic, you know the these equations represent the family of characteristics or in the equations are called characteristic equations. From the family of equations, we can actually read off zeta of x comma y and eta of x comma y. I mean, ultimately, we are actually interested in these transformations, zeta of x comma y and eta of x comma y, which can give us you know A capital A equal to 0 and capital C equal to 0, right.

So, then we found that in order to do this, we can find these directions dy by dx is equal to minus r and dy by dx we can solve for this from which we can extract z, the curve the family zeta of x comma y equal to c 1 and eta of x comma y equal to c 2. And the moment we extract these families, immediately we can pull out this transformation. I mean, we really care about the transformation itself more than these families of curves, right.

So, but the thing is the moment we solve for these differential equations, we get these families of curves and looking at these equations for these families of curves immediately, we can actually read off the correct transformation, right.

So, this is sort of best understood with the aid of an example. So, we will look at an example. But the key point is that, if once we have found these transformations zeta and eta, then what we do is we just go back and plug in the original PDE.

And then, we will see that doing this carefully, working out all these partial derivatives involved, and so on, and rewriting the original PDE in terms of a PDE involving these new independent variables zeta and eta, you will get a form which is either like this or like this. And in fact, we can argue that these two are really the same, right. So, all of this is best illustrated with the aid of an example. So, let us look at an example.

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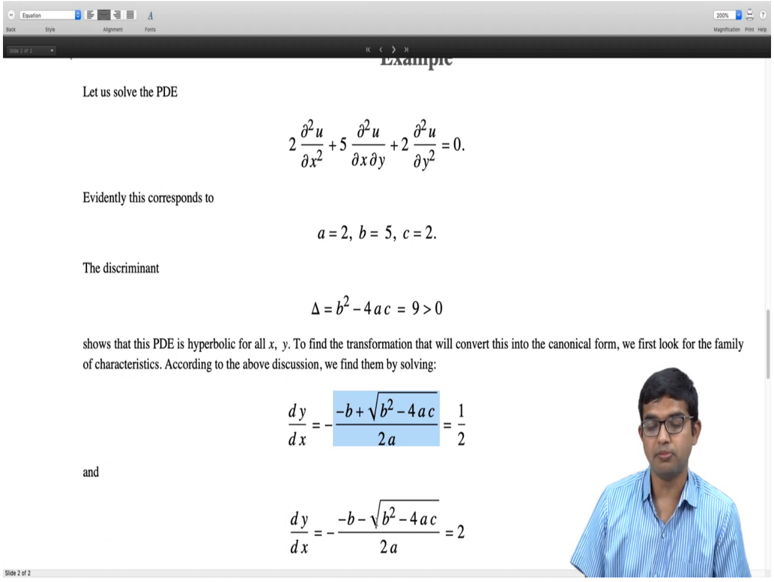

Suppose, we wish to solve this partial differential equation, 2 times dou square u by dou x square plus 5 times dou square u by dou x dou y plus 2 times dou square u by dou y square equal to 0, right. So, this is not a very complicated example, in the sense that there are these terms d, e, f which are already and G, are all 0, and so, we are only focusing on these a, b and c.

So, here we immediately see that a, b, and c are just numbers, they are not even functions of x comma y, right. So, if you look at a equal to 2, b equal to 5, and c equal to 2 the discriminant we see is b squared minus 4 ac is 25 minus 16, which is 9, which is greater than 0, and it is the same at all points x comma y. So, indeed this is a partial differential equation of the hyperbolic type and it is a the of the hyperbolic type for all x and y.

So, now, according to our prescription we must look for the family of characteristics. And in order to find it what we do is we solve for this differential equation dy by dx is equal to minus r, where r is simply given by minus b plus square root of b squared minus 4 ac the whole thing divided by 2 a, which you know you can plug in and see it is just 1 by 2, right. So, b squared is, so this is 9 that gives you 3 minus 5 plus 3 minus 2 divided by 2 a which is 2. So, which is you know the minus sign cancels and so, we are left with just 1 over 2.

And the other characteristic equation is given by dy by dx is equal to minus of minus b minus square root of b v squared minus 4 ac. So, this time you will get a minus 5 minus 3, so that is minus 8 or with an overall minus sign outside. So, its 8 divided by 2 a, 4 by a which is just plus 2, right. So, there are these two differential equations, but both of these are very easy to solve, and in fact, both of them turn out to be straight lines in this case.

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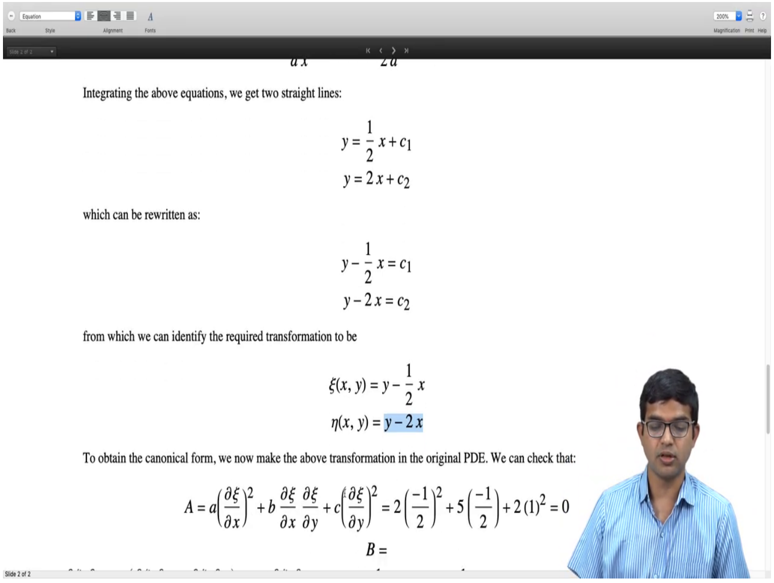

So, for the first one of this is y equal to half x plus c 1 and the second of this is y is equal to 2 x plus c 2. So, but really the both of these are like families of straight lines, right, so which pass through the point x comma y. So, it is convenient to rewrite them as y minus half time x is equal to c 1 and y minus 2 x is equal to c 2.

And once, we see this, so this is of the form zeta of x comma y equal to c 1, that is one family. And this eta of x comma y is equal to c 2. So, from which we immediately get these the transformations which are really what we are after: zeta of x comma y is y minus half x, and eta of x comma y is y minus 2 x.

So, now, if we go and plug back these two transformations in the original PDE, it is going to give us a PDE which will be in the canonical form, right.

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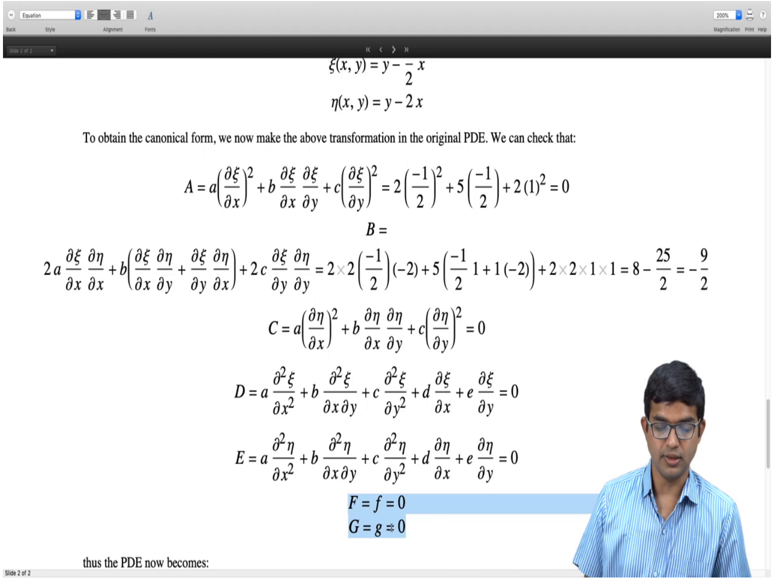

So, let us go back. And in order to do this again we have to compute all these coefficients A, B, C, D, E, F, G. But really, we do not have to worry, it is not going to be all of these coefficients that are not really going to play a role. So, we see that in fact, A is going to be 0 and C is going to be 0.

This has to be the case because that is how we have chosen this A and C, we have taken them to be 0 in our prescription itself. But if you want you can verify this explicitly for this choice of zeta and eta. I mean if you just plug in all of this stuff, I mean I have done this for the first case and it indeed turns out to be 0. You should check it again.

B is of course important, it is not going to be 0, and in this case, I find that the value is minus 9 over 2. You should check this. C is going to be 0. I have taken it to be 0, but you should also check this again. And of course, D, E and F are all going to be 0, in this particular case, because I mean of the nature of this original equation itself, right.

So, even these two guys do not contribute because small d and small e are already absent, and these things will not contribute because dou zeta by; so, the second order derivatives of these are straight lines and when you are taking second order derivatives they are just going to go to 0. So, D does not exist or it just goes to 0, and capital E is 0, F and G are also 0 because the original differential equation PDE we started with itself is not complicated enough to have these other terms, F and G.

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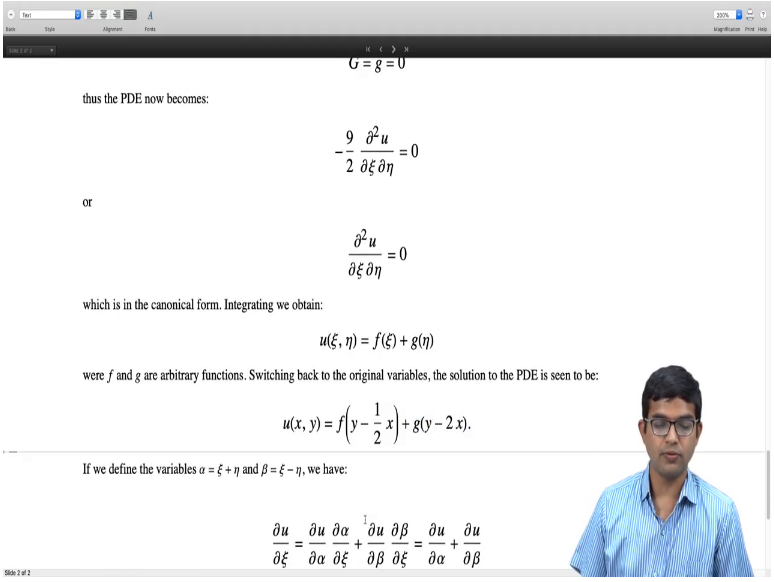

So, now when we do this carefully, we see that there is only one coefficient which has survived and that is just minus 9 by 2. So, the differential equation becomes dou square u by dou x zeta dou eta equal to 0 or which is it is really the same thing as saying dou squared u by dou x zeta dou eta equal to 0, which is in the canonical form.

Now, we can integrate this once, right. So, this is like a constant, and then you can integrate the second time, you know f of zeta comes about you know you can integrate with respect to one of these treating the other as a constant or and then you can do it again. And so, that is going to give you another constant, but you know this constant can be a function of the other variable in each of these.

So, like for example, here if you take a partial derivative or with respect to u, with respect to zeta, you see that the first one will survive the second one is 0, because it is really a constant as far as the first variable is concerned. And then, you take the second derivative, the other one you also go. So, the solution is trivial to write down immediately once we have this canonical form, and where F and G are arbitrary functions.

Now, going back to the original variables, we can immediately write down u of x comma y is equal to f of y minus half x plus g of y minus 2 x. So, that is the power of this canonical form is that we can actually write down the general solution, we can sort of immediately by inspection write it down.

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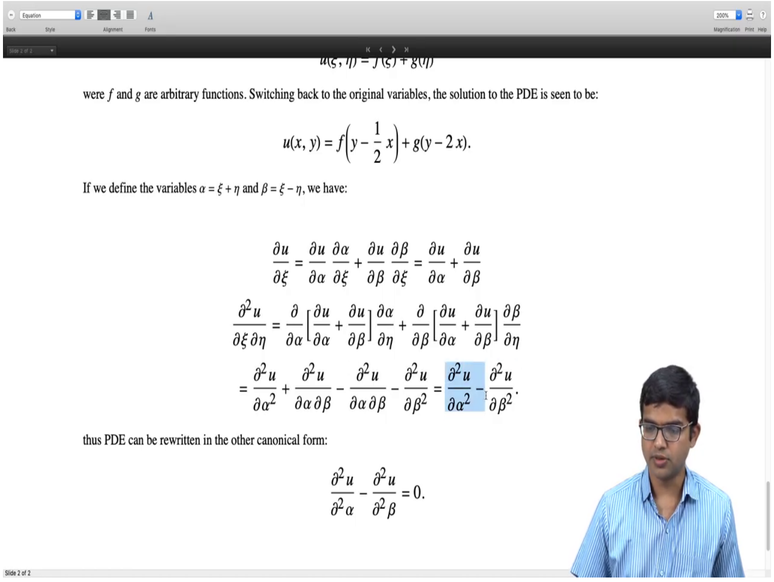

Now let me quickly tell you how if we had used these slightly different variables, alpha is equal to zeta plus eta and beta equal to zeta minus eta. Then, you know this dou u by dou zeta itself can be written in terms of you can check this as dou u by dou alpha plus dou u by dou beta because after all dou dou alpha by dou zeta is 1 and dou beta by dou dou zeta is equal to 1.

So, likewise you can write these dou squared u by dou zeta dou dou eta in terms of all of this stuff, and so, you get dou squared u by dou square dou alpha squared. So, it is going to be. So, let us look at. So, a slightly different way of writing this. So, we should have, should be a little more careful and actually put this 2 here, but it is just a matter of some notation.

So, it should be; so, that is dou beta squared and indeed these two are two terms will cancel and then we are left with just dou beta square. So, which is also, which is the other canonical form. So, let me actually replace this as well. And so, this would be our you know one of the two canonical forms. So, I have said that it can be written in this manner or it can be written in this manner, right. So, this is the form that we just found in our example, and I have also shown you how explicitly one can go from one form to the other ok. That is all for this lecture.

Thank you.