Mathematical Methods 2
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Module - 07
Partial Differential Equations
Lecture - 64
Canonical Form for Elliptic PDEs

So, there is a third type of PDEs in the classification that we have been discussing, and that is the elliptic PDEs, which will be the focus of this lecture. We will work out how to take elliptic PDEs, and convert them into the canonical form ok.

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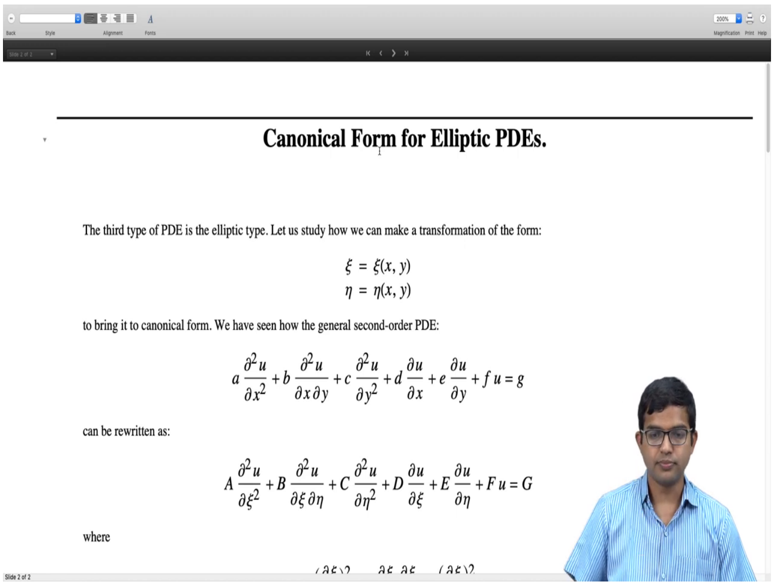

So, we are looking for a suitable transformation of this kind. So, we have these independent variables x comma y. We want to find new coordinates zeta which is a function of x comma y in general, and another coordinate eta which too is a function of x comma y. In general, we start with this second order PDE like here, and go to another second order PDE of really the same kind.

And in particular of course, the relationship between the small a, b, and c will be identical to the relationship between capital A, capital B, and capital C as far as you know the sign of the discriminant is concerned. So, if you are looking at b squared minus 4 ac, it is this going to have the same sign whether you are working with you know small letter variables or capital letter variables right. This is all based on the general discussion we have had.

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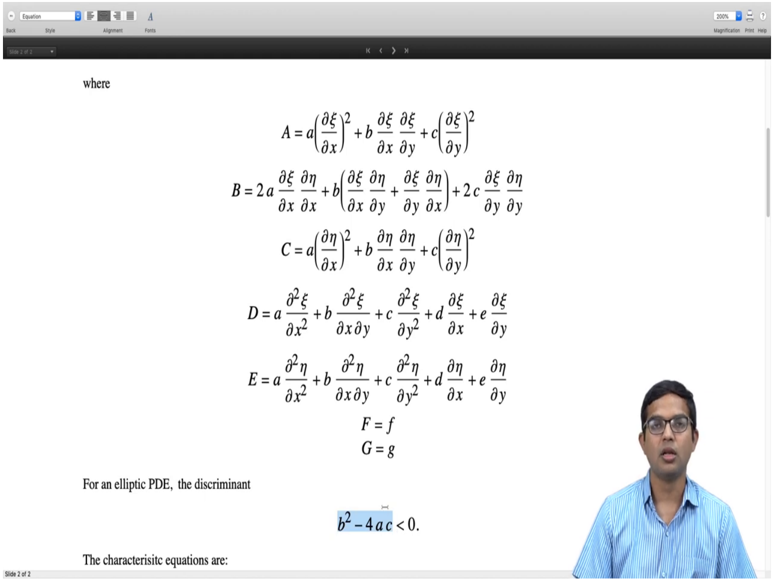

So, and we have also seen that this transformation you know from this form of the PDE to this form of the PDE will are connected by these expressions capital A is given by you know this stuff, capital B is given by this stuff, capital C in terms of this all these partial derivatives we worked out right.

So, it simply comes about from these general transformations we have zeta and eta. And we just go back and plug in using the chain rule, and we go from one kind of partial derivatives to the other. And then do some careful bookkeeping, and then we get these expressions capital A in terms of all this stuff. B, capital B, capital C in terms of small a, small b, small c, again D capital D, capital E and capital F and G of course, these are the two last terms which really remain unchanged.

Now, for an elliptic PDE which is the focus of this lecture, the discriminant is going to be negative b squared minus 4 ac is going to be negative when you are working with these small case variables. And likewise it is also going to be negative if you are working with capital B squared minus 4 times capital A times capital C.

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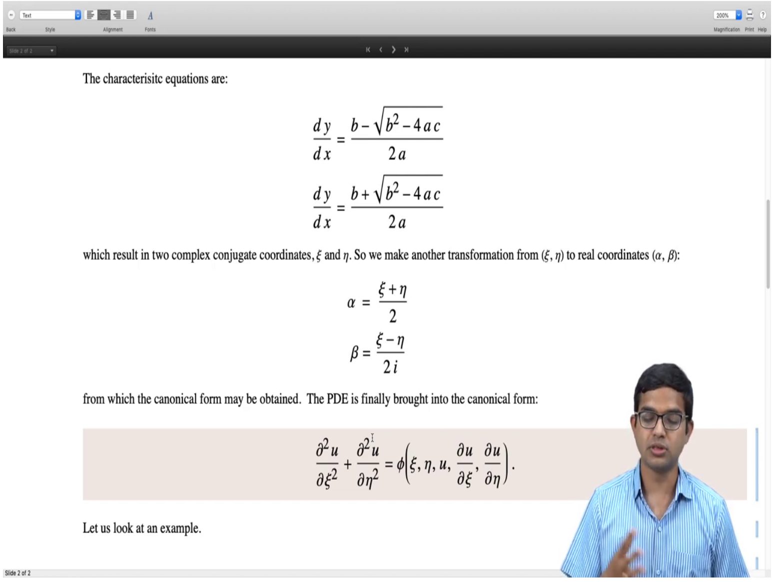

So, the characteristic equations in this case I mean you can you know argue that you have zeta of x comma y is equal to c 1, and eta of x comma y equal to c 2, and then you take this small differential element d zeta or d eta and then you can show that you know the roots of a quadratic equation for which this is a discriminant can get for you the ratio of these of dou zeta by dou x with respect to dou zeta by dou y or equivalently the ratio dou eta by dou x divided by dou eta by dou y.

And from which you know you can argue that there are these special directions at the point x comma y which is given by dy by dx is equal to b minus square root of b square minus 4 ac the whole thing divided by 2 a. And dy by dx is equal to b plus square root of b squared minus 4 ac divided by 2 a. So, once we have you know really understood this machinery quite well, we can actually directly go to the characteristic equations.

So, we know what small a, small b, and small c are. So, we just simply write down the characteristic equations. Solve for these characteristic equations in terms of x comma y, and so then we are going to you know be able to write the solution of these in terms of in the form of some zeta of x comma y equal to c 1, and eta of x comma y is equal to c 2 from which we can read off the transformations that we are after zeta of x comma y, and eta of x comma y.

And in this case particularly we see that b squared minus 4 ac is negative. So, we know from our knowledge of quadratic equations that when b squared minus 4 ac is negative, you are going to get complex conjugate pairs. So, you know these two will turn out to be complex conjugate pairs. And so in fact, which will give us these two zeta and eta will be complex conjugates of each other right. So, all of this is of course best illustrated with the aid of an example right.

So, but before we do that let us quickly point out that once you have these complex conjugate pairs, it actually is convenient to work with real coordinates. So, what we do is, we consider not zeta and eta as it is, but we will consider these linear combinations. So, if you take alpha is equal to zeta plus eta divided by 2 that is going to be a real variable, and then beta is equal to zeta minus eta divided by 2 i right.

So, from which so basically we will make a transformation from x comma y to alpha comma beta not just zeta comma eta when we are working with elliptic coordinates because it is more convenient to work with real variables. So, then finally, the PDE can be recast in this canonical form.

So, you have dou squared u by dou zeta squared plus dou squared u by dou eta squared is equal to some potentially complicated function of zeta eta u dou u by dou zeta and dou u by dou zeta. So, there is no other term involving any kind of second order derivative right. And so the key point is that in the canonical form the second order derivative there are no cross terms here right.

So, I mean we saw in the hyperbolic case, we could have you know either you have only cross terms or you have only non cross you know the sort of so called diagonal terms. And the signs of these two terms were opposite of each other. But for the elliptic case both the signs are the same that is what makes it an elliptic differential equation, it is really the coefficient of sitting here and this here is what decides that this is a of elliptic type. And this the right hand side can in practice be quite complicated.

And you will see that in general we may not be able to just use even the canonical form to write down a full solution right. So, there are you know there is another kind of machinery that has to be brought in after we have the canonical form too.

But the main point of this whole general method is to argue that you know you can take rather complicated looking PDEs and convert them into certain special forms which can be studied further and where we can build our tools. So, let us illustrate all of this with the help of an example.

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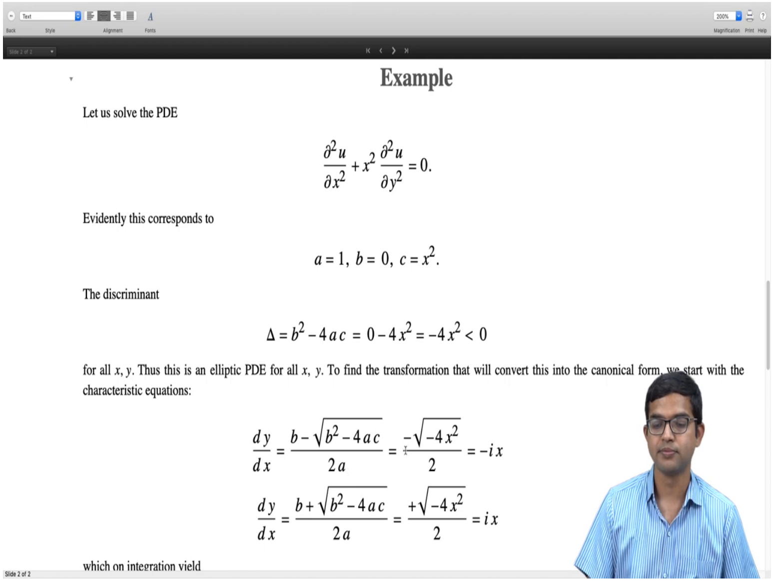

Suppose we are looking at this partial differential equation dou squared u by dou x squared plus x squared times dou squared u by dou y squared equal 0. So, this corresponds to a equal to 1, b equal to 0, and c equal to x squared. So, the discriminant here is b squared minus 4 ac is nothing but b 0. So, it is just 0 minus 4 times x squared, so it is not x squared, y squared, it should be just 4 times x square right.

So, there is no y squared. So, it is a typo. So, it is minus for a is just 1 minus 4 a, and c is x squared. So, it is minus 4x squared. And so indeed for all x and y, this quantity is always going to be negative because there is this minus sign sitting and x squared here is always a positive object 4 times x squared is always positive. So, minus 4 x squared is always negative.

Thus, this PDE is an elliptic PDE for all values of x and y. And so we will be able to find a transformation that will convert this elliptic PDE into another elliptic PDE you know for all x comma y which will go to some other variables. And we will put it in to the canonical form that we just described here. And in order to do this, the starting point is of course, to write down the two characteristic equations.

So, dy by dx is equal to b minus square root of b squared minus 4 ac the whole thing divided by 2 a which in this case is minus square root of minus 4 x squared the whole thing divided by 2. So, this minus square root of minus 4 x squared is really 2 times i times x right. So, we and then these 2 cancel, so we get a minus i x. And the other differential equation in this case is b plus square root of b squared minus 4 ac the whole divided by 2 a, which amounts to just i times x square root of minus 4 is just 2 times i and so you get a plus i x here.

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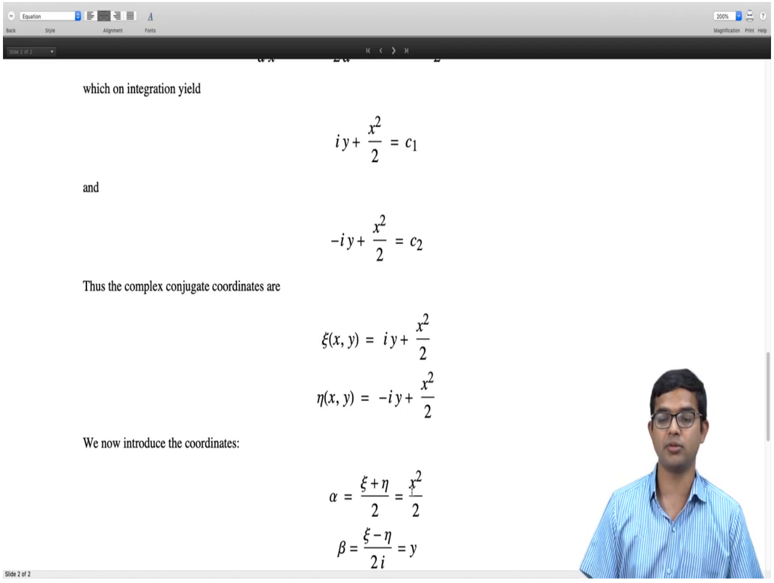

So, there are these two different differential equations which can be immediately solved to yield I mean you what you do is in the first case you pull this i to the left hand side, and then you have i y, and then you send dx to the right hand side. So, you get an x squared by 2. So, i y plus x squared so minus. So, let us see if we get the sign right. So, minus minus 1 over i which is plus i y, and then there is an x squared by 2 ok.

So, one of these will be this equation and the other one is going to be this, you can check this. And indeed if you have two different solutions i y plus x squared over 2 equal to c 1 and minus i y plus x squared over 2 is equal to c 2 so which effectively gives us 2 complex conjugate coordinates. One of them is zeta of x comma y is i y plus x squared by 2. And the other one is eta of x comma y which is seen to be the complex conjugate of this is minus i y plus x squared by 2 right.

So, it is messy to work with complex numbers. So, we introduce new variables which you know like in a in the general description we said you take the linear combinations in this way. Alpha is equal to zeta plus eta by 2 which is x squared by 2, and beta is zeta minus eta divided by 2 i which is just y. So, we work with alpha and beta as our new variables x squared by 2, and beta equal to y.

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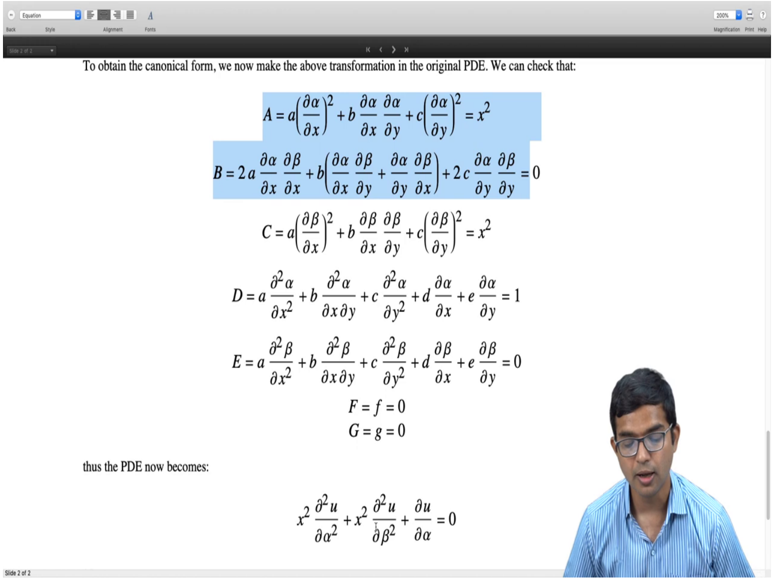

To obtain the canonical form now of course we have to work with dou, so we had zeta and eta earlier. So, in place of zeta, I have put alpha and in place of eta, I have put beta right. So, all these transformation conditions are really the same except that in place of zeta, I put alpha; and in place of eta, I have put beta here.

So, like you can see here. So, when you carry out these computations carefully, you get capital A is equal to x squared, then you get 0, C will again go to x square, D is a is 1, and E, F and G are all 0 right. So, this is something that you should check that indeed this calculation is correct.

And so now, we get our PDE takes this canonical form right. So, there is still one more step left. So, I have x square times dou squared u by dou alpha squared plus x squared times dou squared u by dou beta squared plus dou u by dou alpha equal to 0. But x squared itself should be written in terms of this alpha so which is which gives us another factor of 2.

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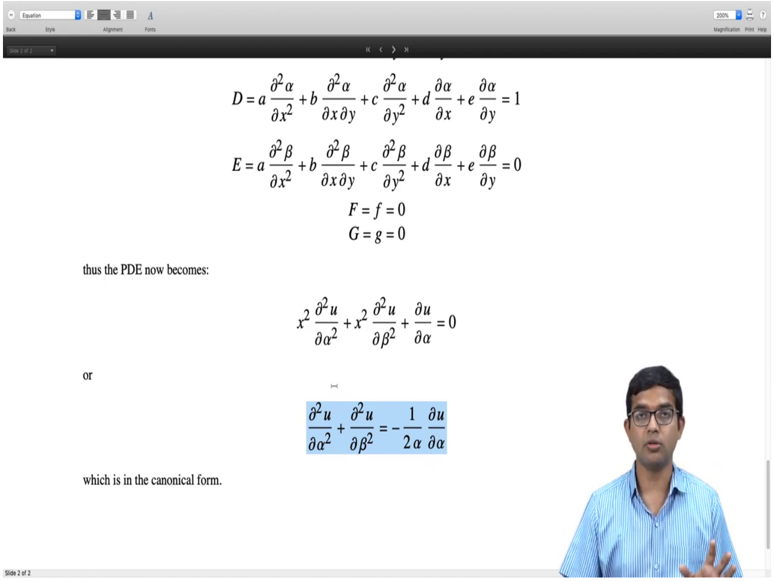

So, x squared is actually nothing but 2 alpha. So, I can rewrite this as dou squared u by dou alpha squared plus dou squared u by dou beta squared is equal to minus 1 over 2 alpha dou u by dou alpha which is really in the canonical form. The canonical form simply means that we have only the diagonal terms as far as the second order differential derivatives are concerned, and both their coefficients are the same, and have the same sign very importantly that is what makes it an elliptic differential equation.

Now, this final elliptic differential equation itself cannot be immediately solved. There is no obvious immediate solution to this right. So, there is a whole machinery associated with solving such differential equations which we will go over in detail, but our descript our discussion so far has been sort of at a very very general level to show that you know you start with some PDEs, and then there is a way to first of all classify them into these three different types.

And there is a transformation in using the method of characteristics by which we can recast them in this canonical form right. So, we will look at each of these different types specifically motivated by physically motivated examples and study how they can be solved in specific contexts involving boundary conditions and so on, but that is coming up later. And that is all for this lecture.

Thank you.