Mathematical Methods 2
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Module - 07
Partial Differential Equations
Lecture - 66
The Laplace Equation: Separation of Variables

So, we introduced the Laplace equation, and we saw some properties of the Laplace equation which we could arrive at from the form of the equation itself. So, in this lecture, we will describe a powerful method called the method of separation of variables by which we can solve the Laplace equation. So, we will work this out in Cartesian coordinates. We are still going to work with functions of two variables x and y ok.

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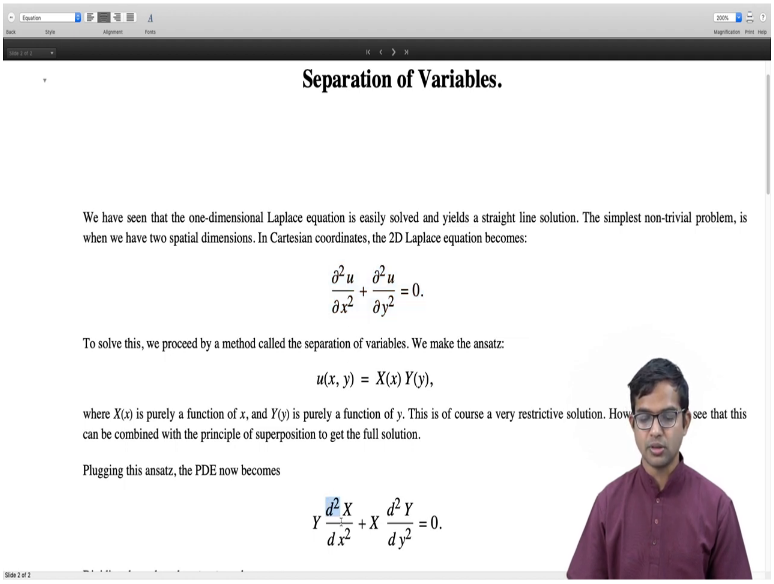

So, yeah, so we have seen that the solution in 1D is of course very straightforward, it is just a straight line. But even from the 1D solution, we could extract some useful properties which we will see will extend to higher dimensions as well. So, the simplest non-trivial problem is when we have two dimensions.

In cartesian coordinates, the 2D Laplace equation is simply given by dou squared, you know u of x comma y divided by dou x squared plus dou squared u of x comma y divided by dou y squared equal to 0. And in order to solve this, we make this ansatz which is called you know within this method called the separation of variables.

So, the ansatz is to you know look for a solution of where you are able to write the you know part which depends on x separately, and a part which depends on y separately and take the product. So, write down your solution as u of x comma y is equal to X of x times Y of y, and then look for solutions of this form.

So, when we are doing this it might appear that we are imposing a and a restriction which is you know which is very artificial. And we may you know be able to find a very special kind of solution alone. But it turns out that you know this separation of variables along with the principle of superposition allows us to.

In fact to stitch together many such solutions and in fact find the general solution of such a problem you know with pretty much any kind of boundary conditions, it turns out it is possible at least for the kind of boundary conditions which are you know physically motivated, this is not just a you know a method to give you some you know idiosyncratic type of solutions. But in fact, it can give you the general solution. There is a prescription which we will describe.

Now, what is the next step? So, the next step is to take this ansatz and plug it into this PDE. So, we have y times this now; you do not have to write dou squared here because now this is purely a function of x alone, x is a purely a function of x, Y of y comes out. So, we have Y times d squared X by dx squared, and again likewise X comes out plus X times d squared Y by d y squared equal to 0.

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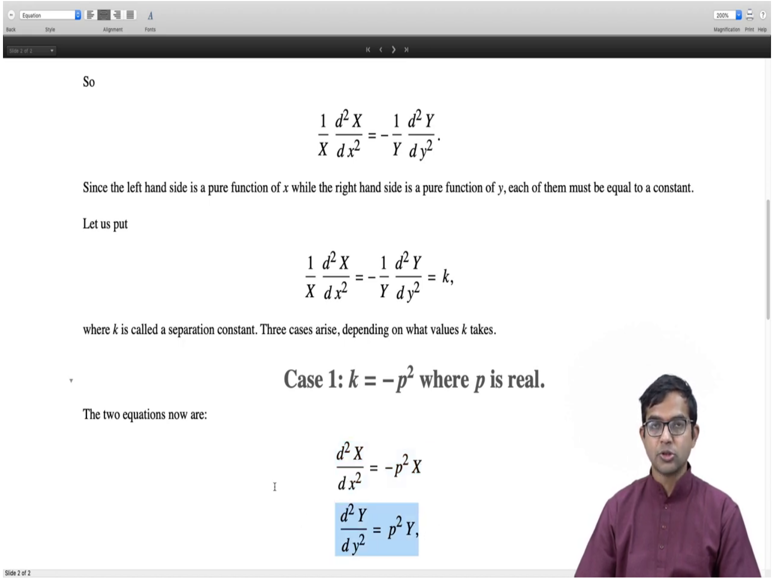

So, if we divide throughout by u of x comma y which is nothing but X times Y, so in the first equation, Y cancels out; in the second one, capital X cancels out. So, we are left with 1 over X d squared X by dx squared plus 1 over Y d squared Y by dy squared equal to 0. So, rewriting this as 1 over X times d squared X by dx squared is equal to minus 1 over Y d squared Y by dy squared is very instructive. So, this is where it is like the core of the method of separation of variables argument.

So, the idea is that now we have managed to write this equation you know where there is only stuff which depends on X on the left hand side, and there is only stuff which depends on Y on the right hand side. So, the only way that you can have some scenario where f of x is equal to g of y where f and g are some functions, and you know the variables involved are x and y.

So, you have the freedom to change x, you can x can take all real values y can take all real values. And if two functions have to be equal, the only way this can happen is if both of these functions are equal to a constant, and both of these are equal to the same constant right, so that is the core of the argument involving you know this method.

So, now, this constant how we choose this constant gives us different kinds of solutions right, so that is where the boundary conditions will come in. But now at this general level of discussion, we can break this down into three different kinds of solutions. So, we have you know we equate this to some constant we call k. Now, depending upon what this k value is, there are three different types.

So, if we take k to be minus p squared where p is real right. So, it is a; it is a negative real number. If k is a negative real number, then we get you know the differential equations involved are these two these are both ordinary differential equations which we are familiar with. We know how to solve these kinds of ordinary differential equations right.

So, this is one reason why we need to have a thorough understanding of ordinary differential equations of you know properties of Fourier analysis and so on before we embark on our study of PDEs. So, d squared capital X by dx squared is equal to minus p squared X is 1. And the other one is d squared Y by dy squared is equal to p squared Y.

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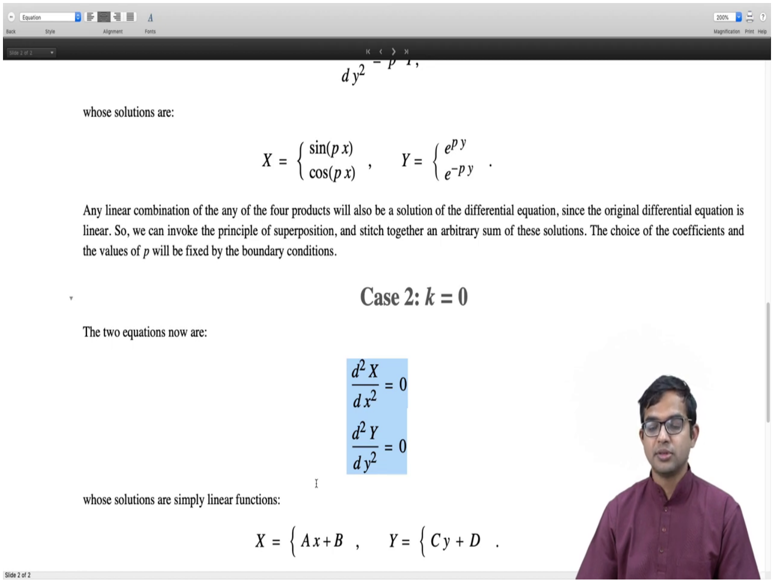

If we solve these, the first of these will give us sines and cosines, sine of px and cosine of px are both solutions of this right. So, I mean this is an ordinary differential equation of order 2. So, indeed we expect two independent solutions right. So, we can write these two independent solutions as sine of p x and cosine of p x. We could also have written it as in terms of e to the i p x and e to the minus i p x, it does not matter.

So, let us write it as sine and cosine, it is convenient. And on the other hand, these independent solutions are e to the p y and e to the minus p y. So, you get exponentials for you know in y. So, the other case is when k equal to 0. If k equal to 0 that is kind of a sort of a you know an in between case and where the differential equations are in are trivial in some sense right.

So, this is the d squared X by dx squared equal to 0, d squared Y by dy squared equal to 0. So, it is really basically a 1D problem of the kind which we already described when we looked at the 1D version of Laplace equation. So, of course, the answer is simply two straight lines: you can write them as A x plus B, and C y plus D.

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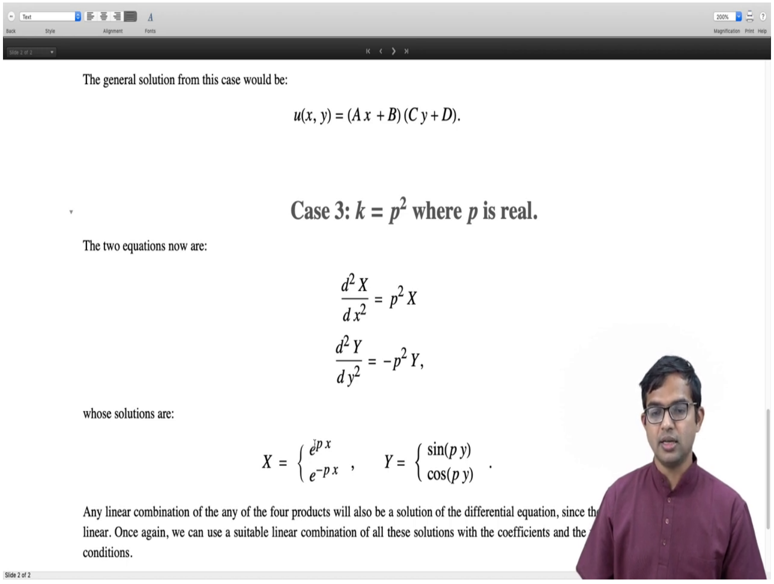

And so the solution would be just simply a product of these two linear solutions right. So, this is you know of a very trivial kind, and perhaps it finds application like very, very special circumstances. Now, the other case is like a mirror image of the first case right. Suppose, we have k equal to plus p squared where p is real, then you get d squared X by dx squared equal to p squared X, but d squared Y by dy squared becomes minus p squared Y.

So, in some sense, this is like reversing the roles that are played by x and y right. So, now what happens is you get exponentials for x and sines and cosines for y. So, capital X is e to the p x or e to the minus p x; and Y the solutions are sine of p y and cosine of p y right. So, the thing is that in general any linear combination of all these four possibilities.

So, both in case 3 and case 2, I mean the solution in general would be of course, e to the p x times sin p y plus some constant times this plus some other constant times e to the p x times cosine of p y plus some other constant times e to the minus p x times sin of p y plus some other constant times e to the minus p x cos p y. So, all four combinations are acceptable, they are all legitimate solutions.

And in fact, the value of p itself can be changed right so depending upon the boundary conditions. So, we will see that imposing you know boundary conditions we will be able to restrict the value that p takes you know. And in some cases case one is more appropriate if these are all you know restrictions which come about from the boundary conditions right.

We will discuss some standard kinds of boundary conditions where you know which are of a great importance in real world examples. And so there we will see that depending upon the kind of geometry that you have the kind of you know behavior you want for large x or small x or certain values of x or and certain values of y you know so one or the other kind is more appropriate, and then also we will have to restrict the values that p takes.

And then even after restriction you may still have an infinite number of possibilities for p. And so we will have to combine coefficients combine all these solutions for different values of p and come up with some in an arbitrary superposition of all of these is going to be, of course, a solution, but then if all the boundary conditions must be satisfied exactly then we will see that all these coefficients also get fixed in a very precise manner right. So, that is coming up in the next lecture. So, that is all for this lecture.

Thank you.