Mathematical Methods 2
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Module - 07
Partial Differential Equations
Lecture - 68
The Laplace Equation in Cartesian coordinates: An Illustrative Example

Ok. So hello everybody. We have been looking at solving the Laplace equation using the method of separation of variables. And so, in this lecture we will look at an illustrative and interesting example where we apply the methods that we have already developed and you know show you in detail how it works out in practice, ok.

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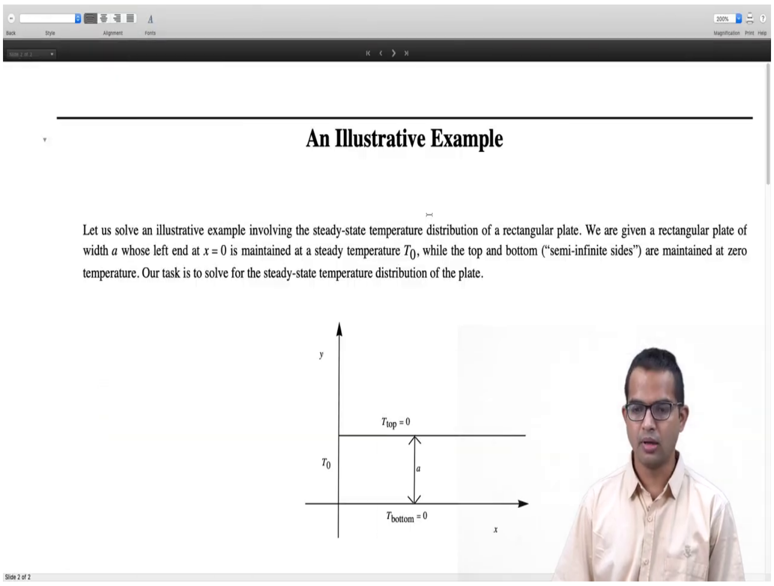

So, let us look at a steady temperature, steady state temperature distribution problem. I mean it is really the Laplace equation. So, we have developed it, you know, in a sort of more abstract formulation and it is applicable to solving for the potential in an electrostatic problem, but it is also equally well applicable to find the steady state temperature distribution.

So, imagine this semi-infinite rectangular plate, right. So, the top and bottom are maintained at 0 temperatures of this rectangular plate. So, it lies in the x y plane. So, it has its width is a and so, the left end is maintained at some temperature T naught. So, the question is if this entire system you know reaches steady state, what would be the temperature at you know all the points on this semi-infinite rectangular plate.

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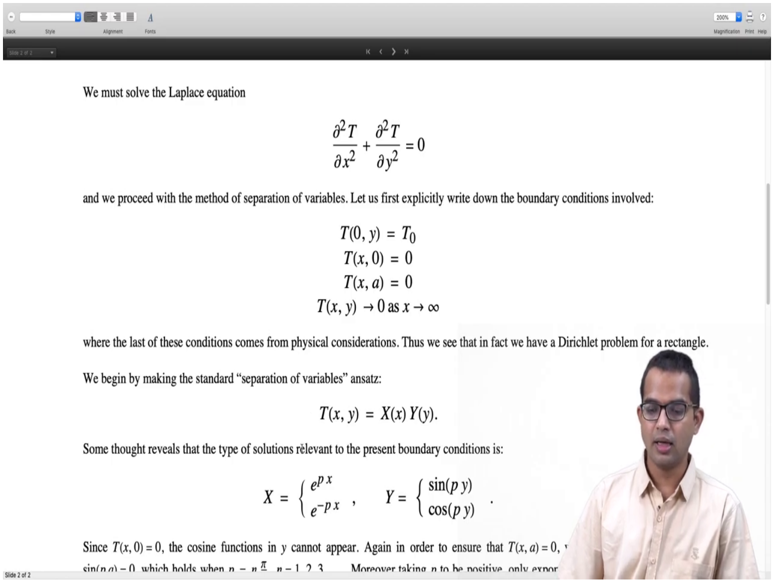

So, the way to solve this is to basically solve for the Laplace equation, right. So, the Laplace equation is dou squared T by dou x squared plus dou squared T by dou y squared equal to 0. And we proceed with the standard method of separation of variables. So, let us quickly write down the boundary conditions involved.

So, the boundary conditions involved here are you know T of you know when x equal to 0 and for any value of y. So, it has to be T naught. That is the left end. Well, I mean any value of y less than a, it will have y is assumed to be restricted to the region 0 to a and t of x, 0 is 0. That is the bottom end and T of x, a equal to 0 that is the top end.

So, in addition to these three boundary conditions, there is also one more boundary condition which comes from; it is a physics you know physically motivated boundary condition, right. So, that is this one which is basically for large x, this temperature must die down, right. So, there is no way to sustain some you know some finite temperature as you go far away because all we are doing is maintaining this end at T naught. So, for T of x, y as x tends to infinity you know in this region, it must die down to 0.

Now, this is once we have these four boundary conditions explicitly written down, we see that this is really a Dirichlet problem, right. So, it is like the boundary conditions are specified on all the boundaries. I mean you have a closed interior for which you want to find or solve the Laplace equation and so indeed there is a solution and that is a unique solution, right. So, there is a uniqueness theorem which we did not go into the details of, but really if you can find a solution that is going to be unique. So, the way to do this is you know the tech technique is the separation of variables.

And we make this ansatz, we look for solutions of this very special form T of x, y is equal to x of x times y of y. And then, we will see that there is a way to stitch together solutions of this kind to match your boundary conditions. And if you are able to find one solution which satisfies the Laplace equation and also satisfies this Dirichlet boundary conditions, then you are guaranteed that is the solution. So, that is where the power of this method comes from.

So, we will recall that if you have a; I mean we could choose exponentials or sinusoids, right. So, in this case we want our solution to die down to 0 for large X and so, it is actually it is appropriate in this context for capital X of x to take on exponentials and capital Y to take sinusoids and cosines because after all at the bottom end and at the top end, it has to go to 0, right.

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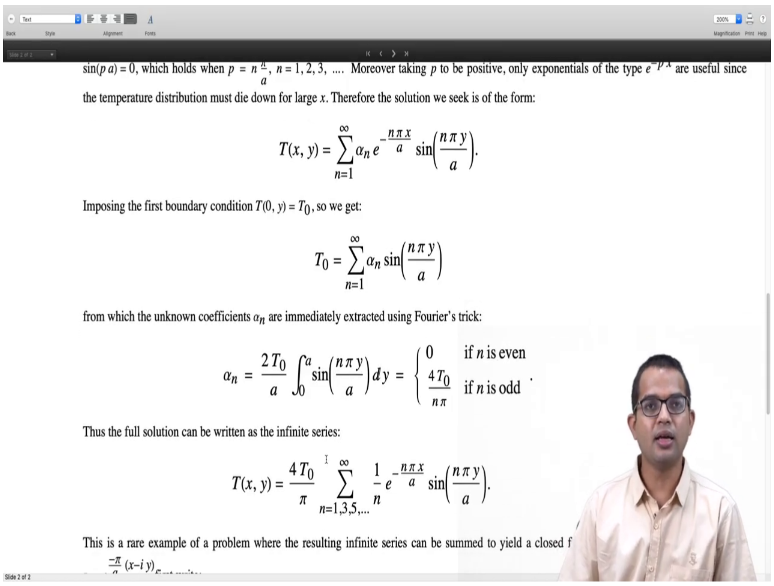

So it is, so it is appropriate to choose sine of p y and cosine of p y for y. The function y and x is going to be exponential.

Furthermore, since T of x, 0 is given to be 0, the cosine functions cannot appear. So, it is in fact only the sine which will appear and also, these p's themselves are restricted, right. So, this is you know like we described in the more sort of general discussion we had. So, p must be in fact we chose to be such that sin of p a must be equal to 0, right. So, that is this boundary condition.

And therefore, p must be chosen to be n times pi over a, where n takes values 1 2 3 so on; n positive integer values. I mean there is an overall constant that you will anyway you are allowed to put in and so, therefore you do not have to consider the negative value. So, we have n taking all positive values.

And only exponentials of the type e to the minus p x will count. So, e to the plus p x is going to blow up at plus infinity which is unphysical. Therefore, we need to work with only exponentials of this kind e to the minus p x. And so, the temperature distribution is then seen to be you know expressible in this infinite series, right.

So, summation over n equal to 1, there are these coefficients which need to be determined right. So, there is still a boundary condition that we must plug in. Alpha n times e to the minus n pi x by a, so that is where you know the constraint on p is put in sin of n pi by a times y.

So, this is the infinite series you know representation for t of x, y and there is still alpha n which needs to be determined. And to do this, we bring in the final boundary condition which is actually the first boundary condition here which is T of 0, y is equal to T naught. So, the left end is maintained at T naught. So, T naught must be equal to when you put x equal to 0, you have alpha n times sin of n pi y by a.

So, this can immediately be worked out using Fourier's trick. So, you just multiply on both sides with the appropriate sin of n pi y by a. If you are interested in extracting this coefficient alpha n, this is the standard technique. So, you can check this and convince yourself that indeed this will extract for you alpha n, right.

So, all the other sines corresponding to other terms will go away when you do this integral, only one of them will survive. And then, you know the factors are appropriately put in you can check it. And then, this integral itself is straightforward to work out and you will get 0 whenever n is even and you get you know this answer; 4 times T naught divided by n pi. So, that will be a factor of a which will cancel with this a and you will just get 4 times t naught divided by n pi if n is odd.

So, basically it falls off with n as 1 over n and then, there is this factor 4 T naught by pi. So, the full solution can be written down. Now we have the full answer. Now basically 10 or T of x, y is equal to 4 T naught divided by pi summation over only odd values of n 1 over n e to the minus n pi x by a sin of n pi y by a. It is a fairly straightforward problem and there is this infinite series solution.

So, what makes this problem particularly interesting is that in fact there is a closed form expression available, right. So, there is a very nice final answer which we can write down if you do a little more work and that in fact connects back to our study of complex variables.

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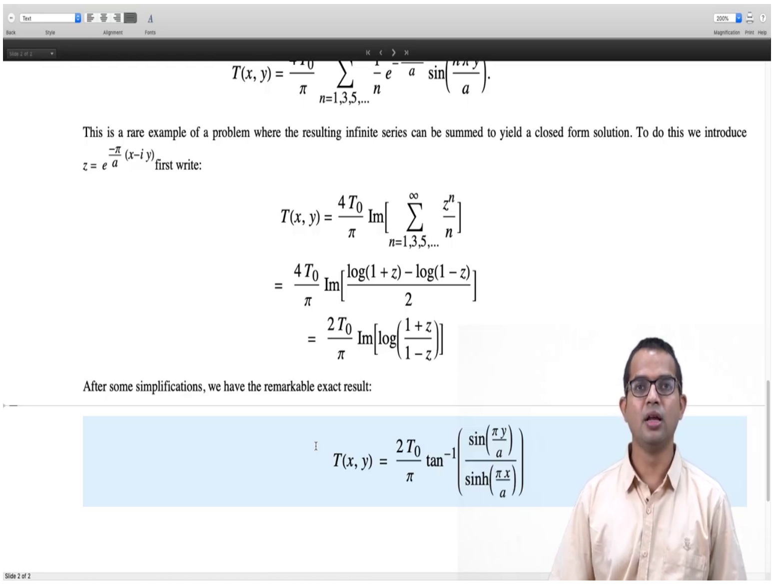

So, the way to work out this infinite series is to look at this quantity z. So, you have sin of n pi y by a. So, whenever you have a series in sines, it is convenient to also consider the same series with cosines instead of sines. And then take cosine you know the term involving cosines plus i times the term involving sine and then, you know conveniently write down cos theta plus i sin theta is e to the i theta and then, you will have these exponentials.

So, I will allow you to check this. So, if you introduce this variable z is equal e to the minus pi a; pi by a times x minus i y, we can rewrite this infinite series as actually nothing but T of x, y is equal to 4 T naught by pi times the imaginary part of this simpler looking infinite series.

So, there are only the odd numbers appearing here n equal 1 3 and so on. Therefore, this series is then seen to be actually a combination of log of 1 plus z and log of 1 minus z. So, the even terms have to be cancelled out and then, you have to take this factor of 2. So, again I will allow you to check this. So, it is the imaginary part of log of 1 plus z minus log of 1 minus z the whole thing divided by 2. And then, this is nothing but I mean you get a 2 T naught divided by pi imaginary part of log of 1 plus z divided by 1 minus z.

So, I will allow you to do some simplifications and then, we have this final remarkable exact result which is T of x, y is equal to 2 T naught divided by pi tan inverse of sin of pi y by a divided by hyperbolic sin of pi x by a. So, it is a very nice, cute result which comes about and it is a rare phenomenon that you know this kind of a method of variable separation of variables will yield for your closed form solution.

And also, in this form it is straightforward to directly plug this in into the original p d, the Laplace equation and check that indeed it satisfies the Laplace equation. Also it is straightforward to check that indeed all the boundary conditions hold. So, because there is a uniqueness theorem, indeed this is the solution. Ok, that is all for this lecture.

Thank you.