Mathematical Methods 2
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Module - 08
Partial Differential Equations
Lecture - 77
The diffusion equation with Dirichlet and Neumann boundary conditions

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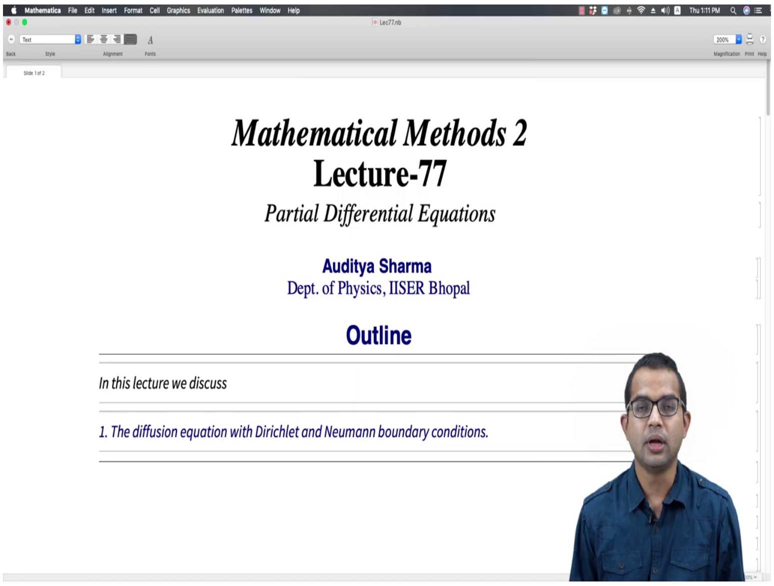

So, in this lecture we will solve the diffusion equation again, but with two different kinds of boundary conditions namely the Dirichlet and Neumann boundary conditions.

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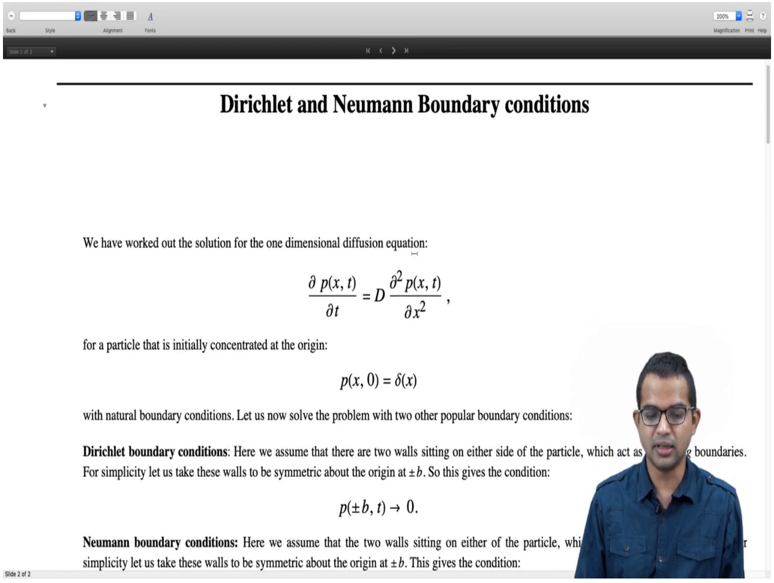

So, the diffusion equation is given by dou p by dou t is equal to D times dou squared t by dou x squared in 1 d and we have seen that if the particle is initially concentrated at the origin and if with natural boundary conditions the solution is a Gaussian right. So, we work this out. Now, there are two other kinds of boundary conditions which we also saw in the context of the Laplace equation, which are interesting, namely the Dirichlet and the Neumann boundary conditions.

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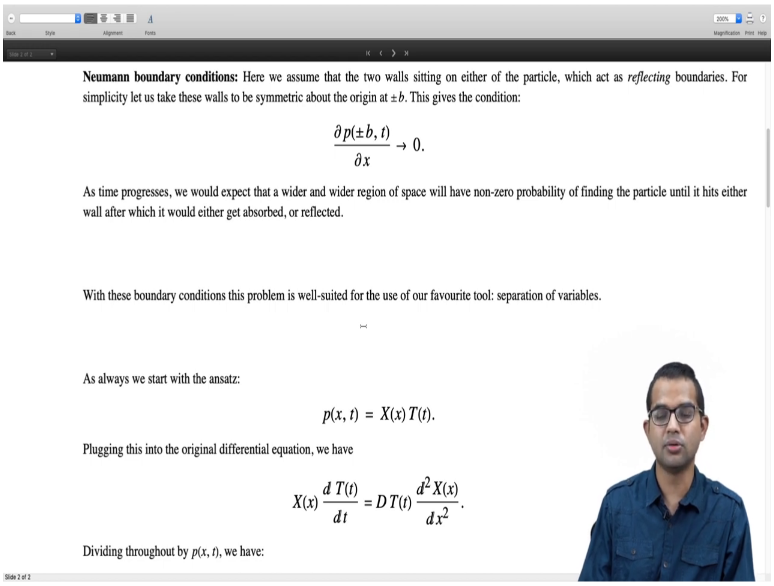

So, in the context of the diffusion equation; the Dirichlet boundary conditions are also called absorbing boundaries. So, for simplicity let us say that you know plus b and minus b there are symmetric walls and you know the job of these walls is just to absorb particles. So, there are particles which are sort of jiggling around and whenever they hit either plus b or minus b they are going to get absorbed, so that is the Dirichlet boundary condition.

So, which means that the probability of finding a particle at either plus b or minus b is 0, that is what you have to put in your boundary conditions. On the other hand, Neumann boundary conditions are also called reflecting boundary conditions where you know whenever the particle hits either plus b or minus b it gets reflected. So, here, what happens is that the particle current goes to 0.

So, at plus b and at minus b the particle current is 0, which in turn implies that dou p by dou x is going to go to 0. So, like we saw when we were deriving the diffusion equation, you know one of the fixed laws tells you that the particle current is actually proportional to the gradient.

So, it's dou p by dou x which goes to 0 at plus b and minus b. So, we will solve both of these problems. We will work out the solutions for the diffusion equation within both of these setups using the method of separation of variables. So, the ansatz that we start with is p of x comma t is equal to X of x times T of t. We separate the spatial and the time variables, if you plug this into the original p d e, we have x times d T by dt is equal to D times T times d squared X by dx squared.

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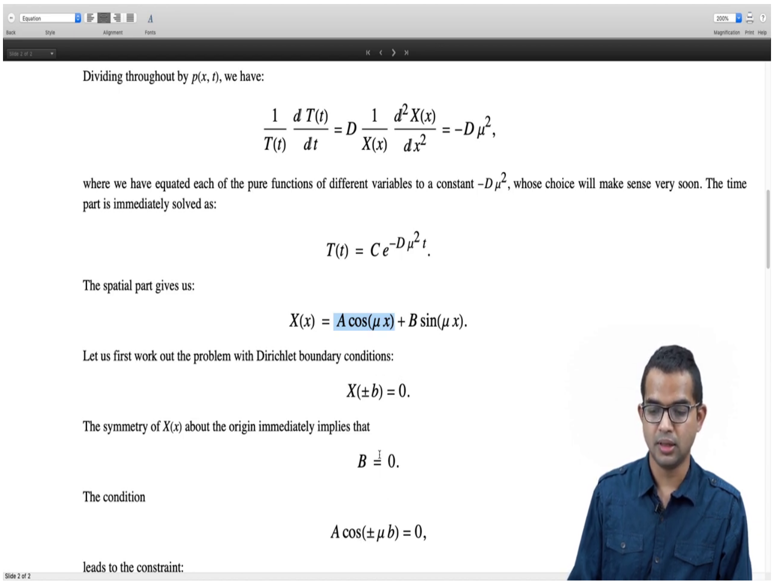

Now, if you divide throughout by p of x comma t, in the first on the left hand side; x cancels, we have 1 over T times dT by dt is equal to D times 1 over X d squared X by dx squared. Now, of course, we argue that you know the left hand side is a pure function of t alone and the right hand side is a pure function of space alone.

So, it must be a constant, and that constant is conveniently written as minus D times mu squared. We will see how this makes sense in a moment. And, with this choice of the constant, the spatial part is the time part is immediately solved right. So, we can see that T of t is given by just some constant time e to the minus D mu squared t.

And the spatial part you know you can get cosine solutions and sine solutions cos of mu x and sin of mu x are both solutions. If we work out the solution with Dirichlet boundary conditions X of plus or minus b must be 0 and the symmetry of this problem right. So, it implies that you cannot have a sinusoidal solution. So, the problem is symmetric so the solution is also going to be symmetric. So, we will look for solutions which are cosine in nature.

Now, we also need to impose this condition that, when X is equal to plus or minus b, we must have you know X of x x of plus or minus B must be 0 therefore, A times cos of plus or minus mu b must be 0 which in turn implies that mu b is constrained to be an odd multiple of pi by 2 right.

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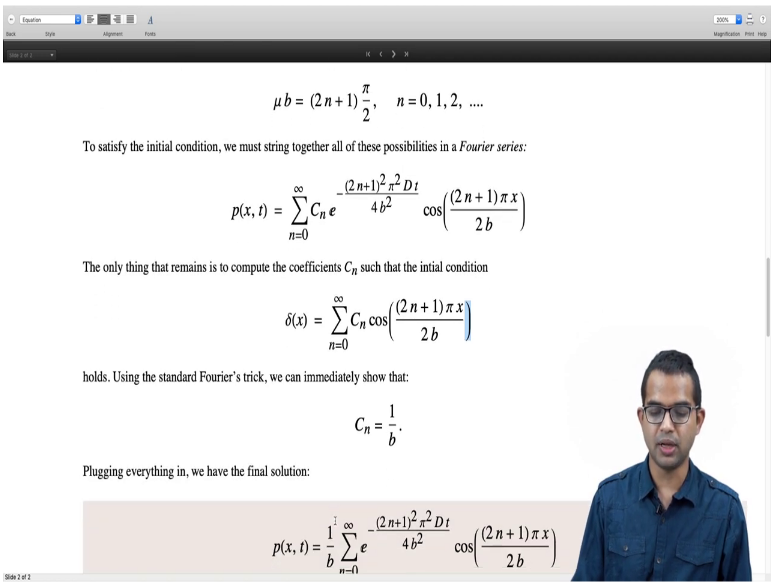

So, therefore, to find the full solution. We must string together all these solutions and write it as this infinite series which is really a Fourier series right. So, the coefficients will be worked out with the aid of the initial condition. So, we have this Fourier series involving only you know odd numbers.

So, we can write this as n equal to 0 to infinity, but 2 n plus 1 the whole squared pi squared dt comes in divided by 4b squared. So, this comes from the condition on mu and then we have only cosine term, because of the symmetry of this problem the way we have set it up and then all that remains is to find out C n, and that is given in terms of the initial condition.

So, the initial condition is just delta of x the particle is localized at the origin. So, summation over n going from 0 to infinity, this whole stuff must be equal to delta of x using the standard Fourier trick, we can immediately show that, in fact, C n must be just 1 over b right. So, you can check this, you just multiply with an appropriate cosine and integrate in this entire interval and you see that you know you can pick the coefficient of your choice and multiply by appropriate cosine function and every other term will vanish and C n you can choose just 1 over b.

So, if you plug in everything we have the solution p of x comma t is equal to 1 over b summation n equal to 0 to infinity this stuff there is an exponential decay term and then there is a cosine term right. So, exponential decay is suitable because I mean, this is going to die down for large T and that is why this choice of minus mu squared here makes sense right. So, I mean we have found a solution for a Dirichlet problem.

So, the boundary conditions were satisfied and p of x comma t does satisfy the differential equation and indeed, so therefore, it is the solution right. So, this is a consequence of the uniqueness theorem now there is this, so we might ask what happens when you make b very large. Should this solution not go back to the gaussian solution that we already discovered. And indeed, this is true, but there is some subtlety involved here right.

So, you see that when b becomes arbitrarily large, you know when here also you have a b becoming very large, so this term is actually just 1 basically. Cosine of 0 is just 1 and on the other hand this term also does not contribute. So, basically you will have an infinite series where all the terms must be kept track of, you cannot truncate this series.

So, in fact, it turns out that it is in the limit of t becoming very large, that is where it is easy it is easier to show how this solution becomes you know like the gaussian solution that we have already seen. So in fact, so you know when t is very large, you see that these exponentials start decaying and you can truncate your series and then quickly check it.

So, there is no simple way to show this. There are more complicated ways of you know analytically showing, that indeed, this will go to the gaussian solution, but we will not go there.

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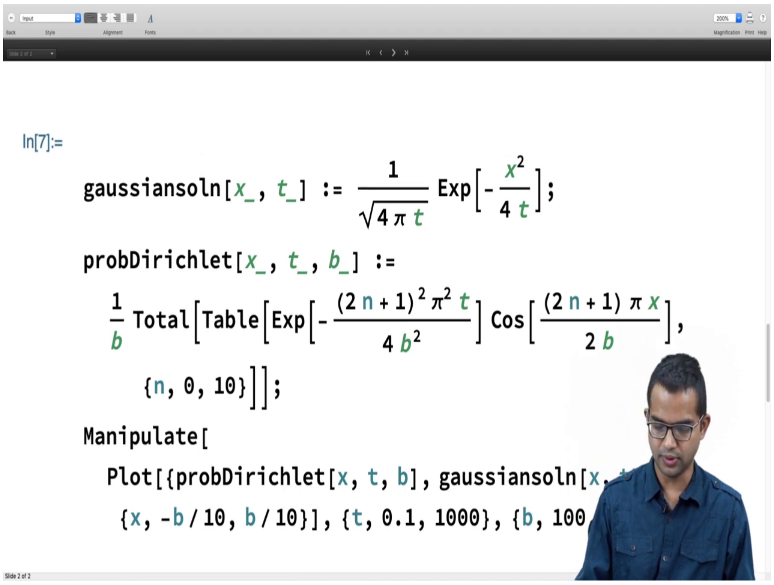

Let us look at a let us look at plots of the gaussian solution that is just this I have taken D to be 1 and then what I am doing here is I am truncating this series after just 10 steps n is allowed to go from 0 to 10, and then I have just you know allowed this parameter b to change and then I have plotted this. So, let me just go ahead and do this and then I will show you what it looks like.

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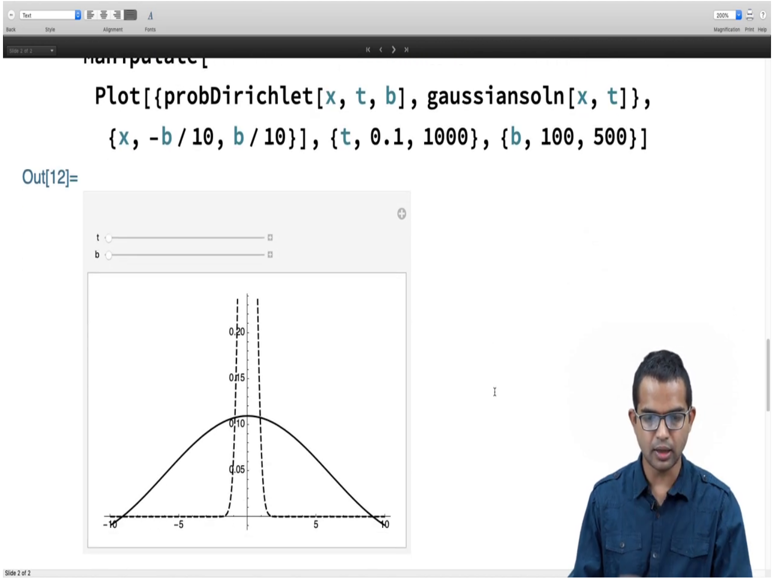

So, if I run this, there you go. So, you see that for you know I have these two different parameters t and b.

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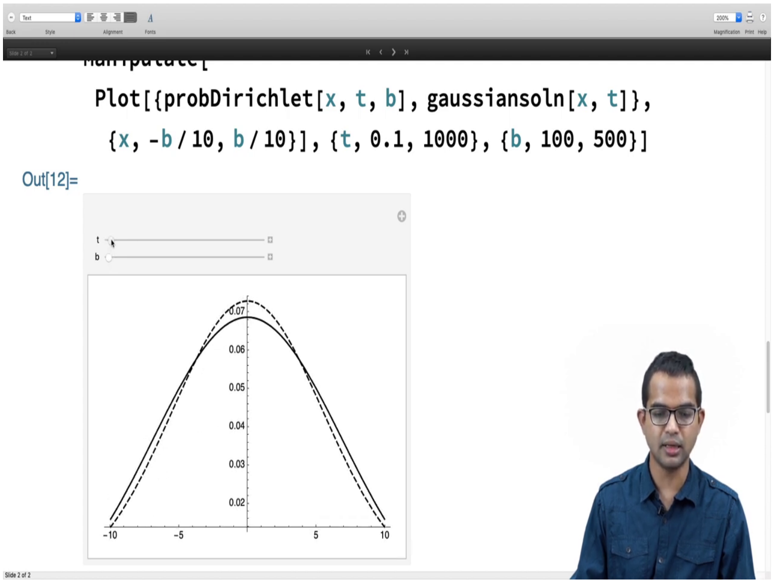

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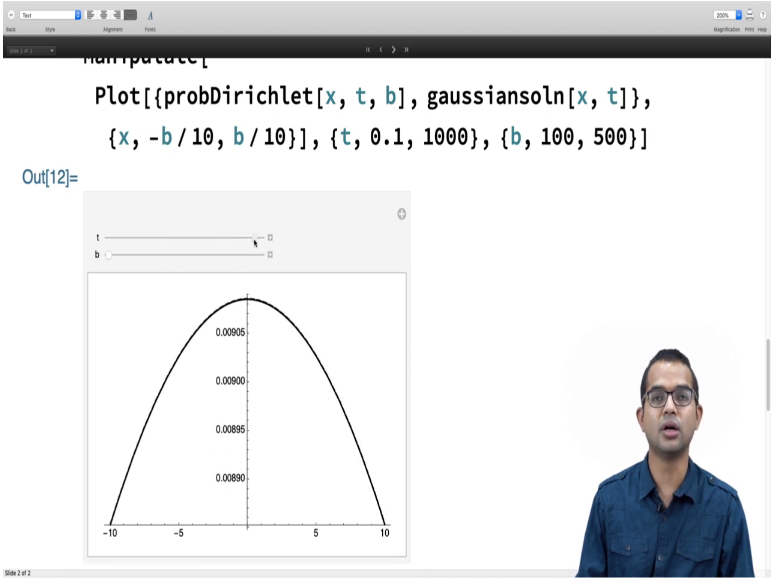

So, let us keep b as it is, and then as I increase t. So, I have these two different solutions; one of them is the gaussian solution and the other one is you know this solution which is coming from the series. So, you see that for law, when I increase t beyond a point the two curves overlap entirely right.

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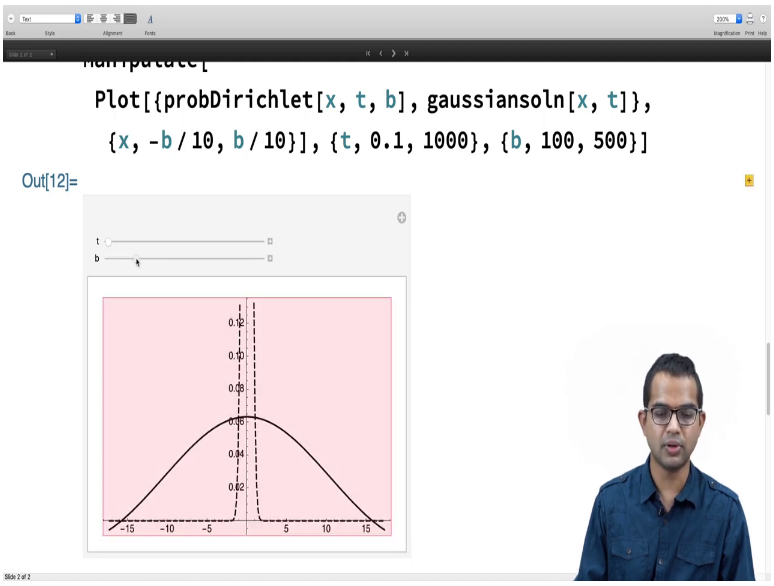

And on the other hand, when I increase b also this will happen, but now I will have to go for longer times before the two of them start merging. So, if I look at very large b then I have to actually keep on increasing time to a much larger value, because I am truncating this series at 10 right.

So, you know, you have this 1 over b sitting outside, but then you have a very large number of terms. We have to add all of them up and then divide by t, you cannot just keep a few terms. So, when b is larger, if you go to very large t's that is when this truncation after just 10 steps becomes justified.

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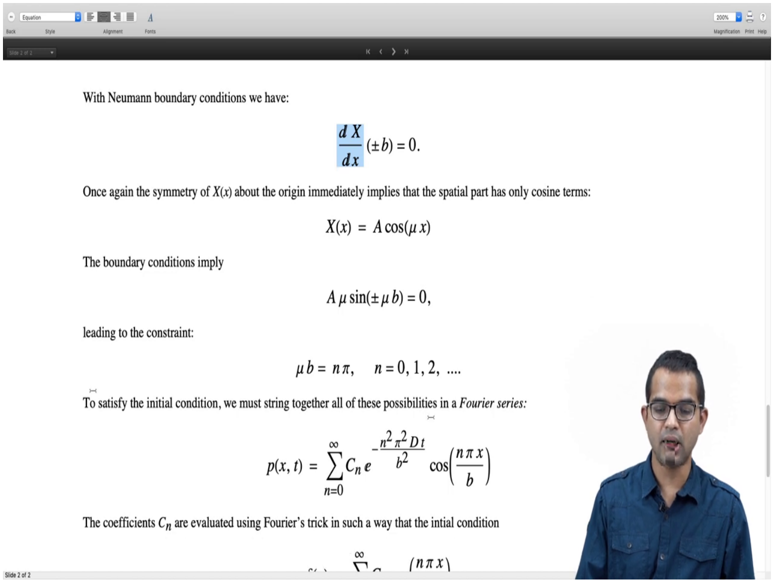

So, likewise we can also solve the problem with Neumann boundary conditions. So, here the boundary conditions are d X by x at plus or minus b must be equal to 0. Symmetry of the problem once again ensures that it is only cosine terms that will survive and then the boundary condition implies here that you know sin of plus or minus mu b is equal to 0. You have to take the derivative and so now, mu b is going to be n pi.

And now the Fourier series is going to run from 0 to infinity C n of this stuff minus n squared pi squared D t by b squared the exponential of this cosine of just n pi x by b.

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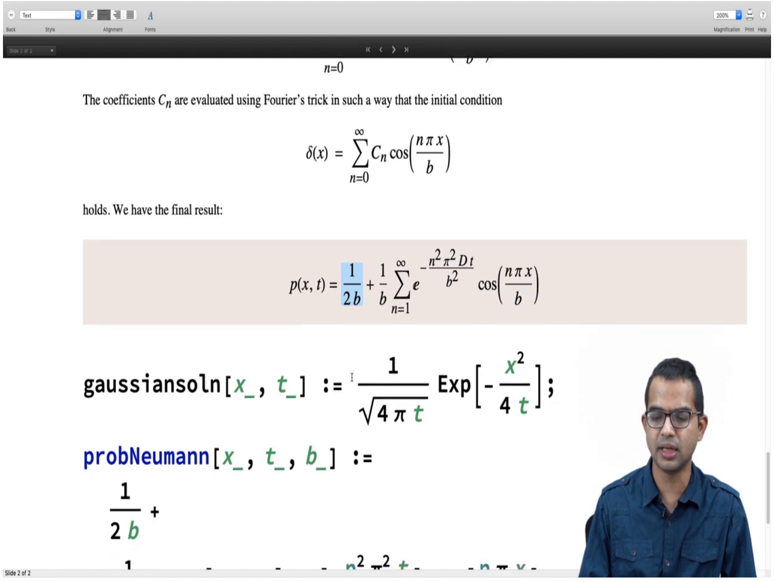

Now, once again the initial condition, we have to plug in the initial condition and evaluate these coefficients, so we have this final result. So, 1 over 2 b comes out in order to evaluate this separately and other coefficients you can evaluate you know in general for other n. You can check this and indeed the solution is going to look like this. There is a 1 over 2 b plus 1 over b times you know an infinite series. So, once again you see that I mean, when b becomes very large this part does not matter.

So, it does not matter whether you have Dirichlet boundary conditions or whether you have 1 and Neumann boundary conditions. Both of them are going to go to the gaussian solution. As we can once again verify, once again I have truncated this series after just 10 points for n equal to 10.

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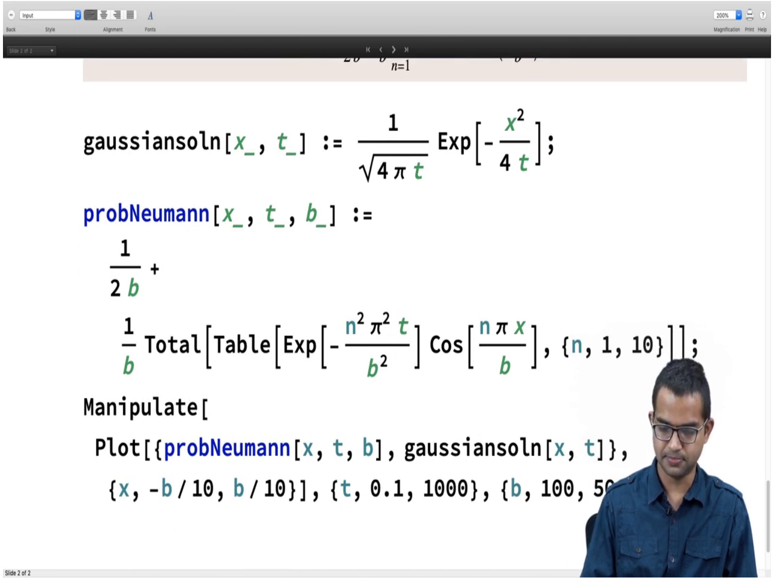

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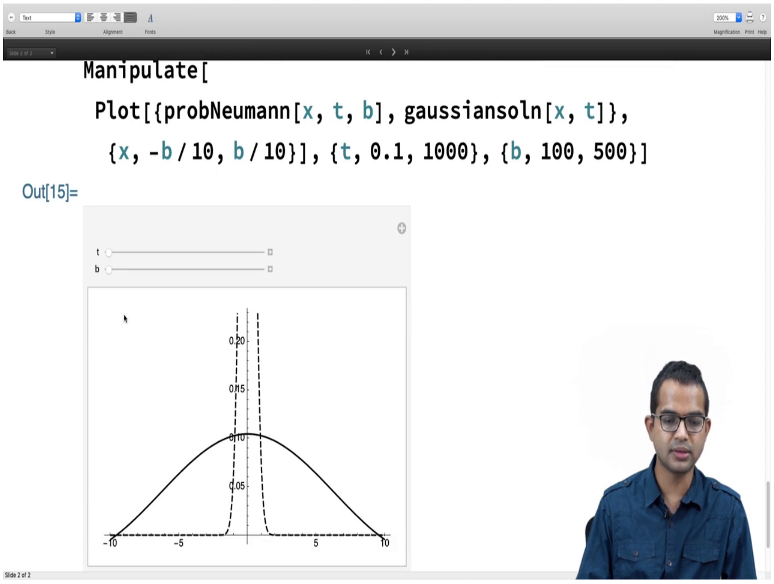

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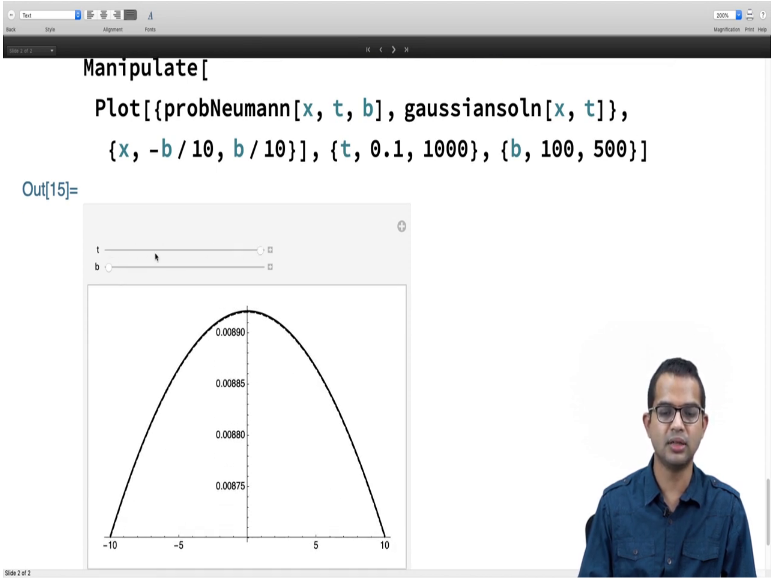

And, so you see once again that if you keep b fixed and keep on increasing t the 2 solutions will merge.

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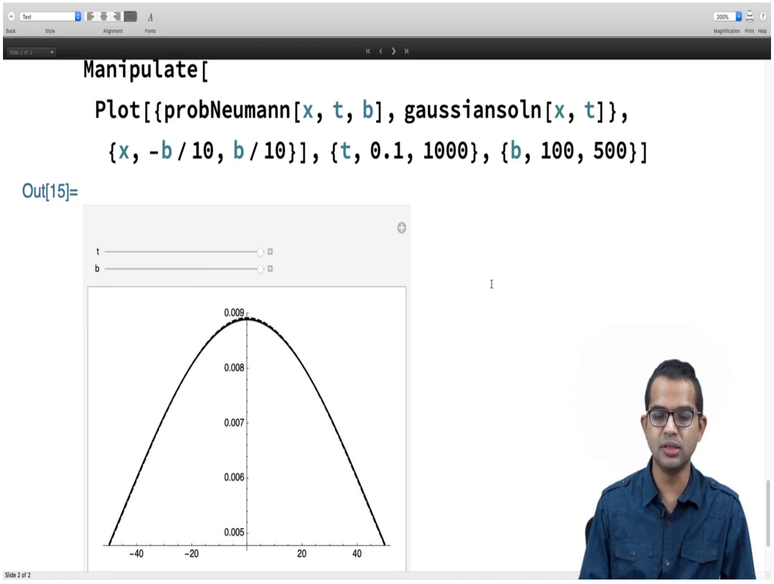

And once again, larger b means that you have to go to much larger t before this becomes justified. So, both these solutions in the with both these boundary conditions are written in this form of an infinite series, where it is more reasonable to think of this you know these are good in the limit of t becoming very large or t by b squared in the becoming large right.

So, if you take you know b to be very large and keep t to be small, then you cannot truncate this series at a small number of steps, you will have to keep a very large number of limits. And, in fact, verifying its equivalence with the gaussian solution is not so straightforward. But in the limit of t becoming very large indeed we have seen explicitly that the solutions agree ok.

Thank you.