Mathematical Methods 2
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Module - 08
Partial Differential Equations
Lecture - 79
The Wave Equation: Method of Characteristics

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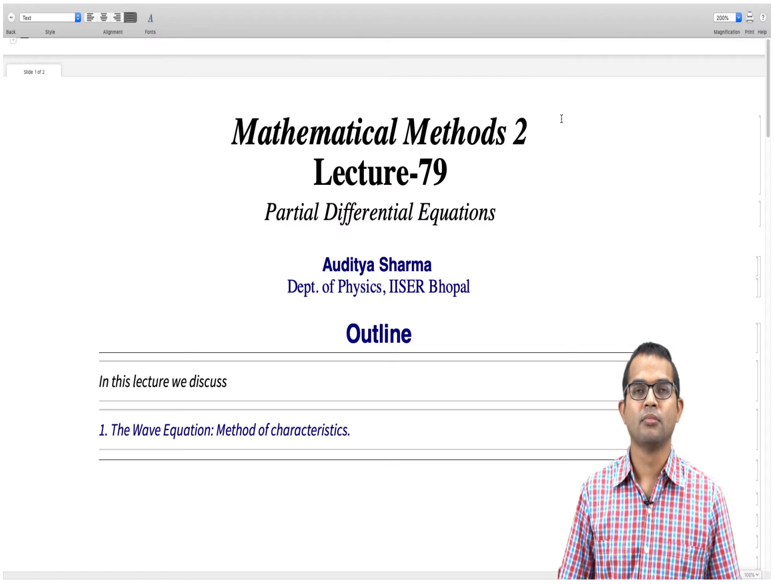

We have studied the Laplace Equation and the Diffusion Equation in a fair amount of detail. So, we will spend a little time on the Wave Equation, right. So, in this lecture, we will look at how to solve the wave equation using the method of characteristics.

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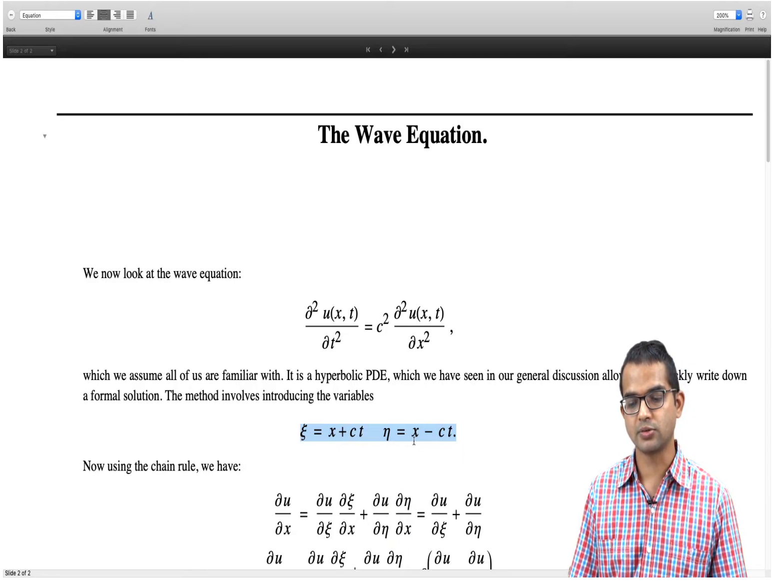

So, when we started from a general perspective discussing PDEs, we classified PDEs into parabolic elliptic and hyperbolic PDEs. And so, we also discussed the method of characteristics in a fair amount of detail, right. So, the wave equation which I assume we are all familiar with, maybe from electrodynamics or you know some of the elementary courses on waves, I mean I assume that all of us have at least seen the wave equation.

So, this is the wave equation which is, this is a second order derivative with respect to time and a second order derivative with respect to x. So, it can be written in a more general form in three-dimensions or you know involving Laplacian operators and so on. But let us consider the simplest version of it in 1D.

So, this is the wave equation, this c, we know has the interpretation of the speed of the wave that we will see again as we go along. So, it is a hyperbolic PDE. So, you can go back and check this discussion of you know classification of PDEs and how this will turn out to be a hyperbolic PDE.

And also, the discussion around the method of characteristics involves finding a suitable substitution, right. So, we will just you know write down the substitution here, right. So, the logic into how to find such a substitution you can go back and find in an earlier lecture.

But, so here, if you just choose zeta as x plus c times t and eta as x minus ct, if you introduce these two variables, and then make this change of variables using the chain rule. So, you have to compute dou u by dou x, which comes out in this form. And dou u by dou t, which comes out in this form, right. So, you can check the algebras, there is just a factor of c and a minus sign in the second of these.

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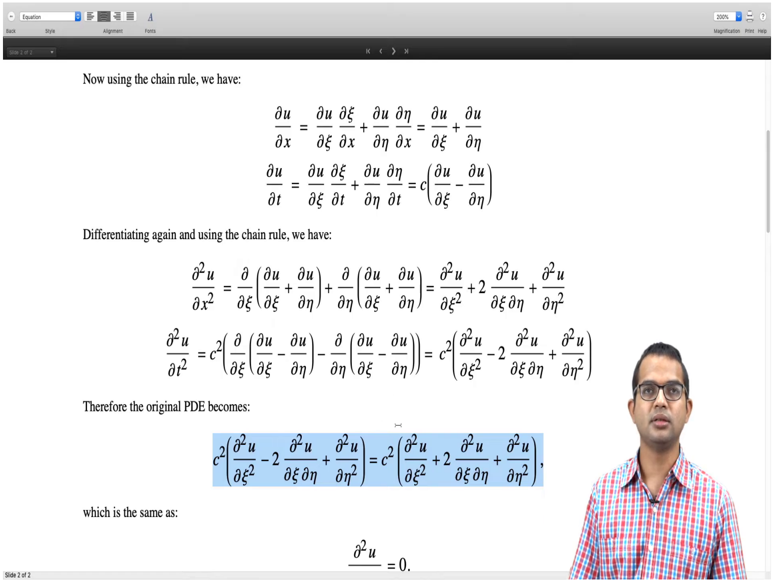

And then, if you differentiate again and use the chain rule carefully collecting terms, you can show that the second order derivative with respect to x, and the derivative second order derivative with respect to t, give us these two expressions. There is a c squared here, and then this is you know it involves these second order derivatives with respect to zeta and eta. So, yeah, basically, these are the two expressions we are after. And so, if you go back and substitute these expressions in the original PDE, so we get you know this equality, and it simplifies quickly to, just, this is a very simple looking equation, right.

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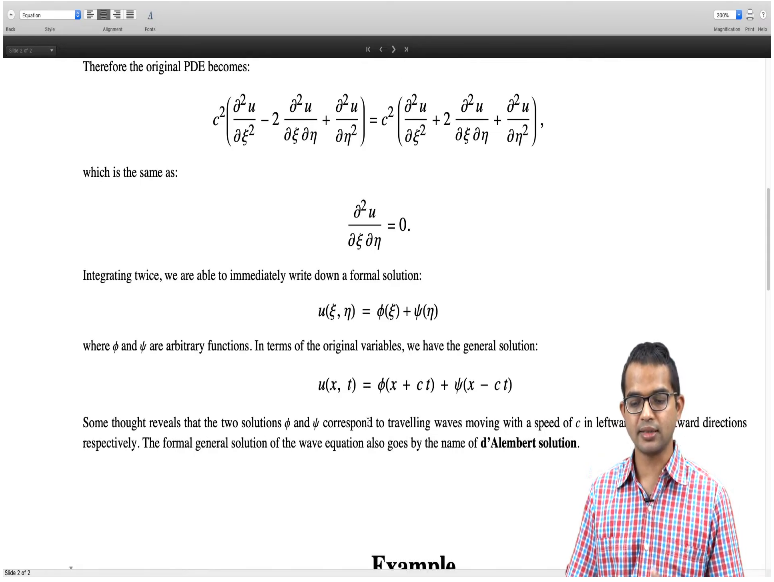

So, that is the whole point of the substitution. The method of characteristics is to find a suitable substitution which eventually converts rpd into an extremely simple form from which in fact, we can read off a general solution. So, the formal solution can be immediately written down. So, u of zeta comma eta is simply some arbitrary function of zeta plus some arbitrary function of eta, right. So, this you can argue by integrating once with respect to zeta, then integrating again with respect to eta, right.

Alternatively, you can just take this and take a derivative with respect to zeta and with respect to eta. So, if you are taking a derivative with respect to zeta, one of them is going to the zeta then psi of eta is going to act like a constant and vice versa. If you are going to take a derivative with respect to eta, this part is going to act like a constant, right. So, that is how this PDE will hold, right.

So, this is indeed the formal solution and going back to the original variables. So, we have this formal solution for the wave equation, right. So, in fact, this is something which is familiar, right. So, we have seen how the wave equation admits these wave solutions. So, this is also known as the d'Alembert solution.

So, some pause, and it reveals that in fact, you know if you take a function of this combination of variables x minus ct, so this represents a wave which is moving in the positive direction. So, the psi of x x minus ct represents a wave which is moving in the positive direction of the x axis at the speed c, while this is a wave which is moving at the same speed c, but in the leftward direction.

Now, this c is a characteristic speed which depends on the medium, right. So, that is already sort of inbuilt into this wave equation for us. So, this c comes from the properties of the medium in which the wave is travelling. But the solutions basically tell us that you know you can either have the wave moving in the right direction or in the left direction, right. So, now, depending upon the boundary conditions and the initial conditions, you know this can be applied to different interesting cases.

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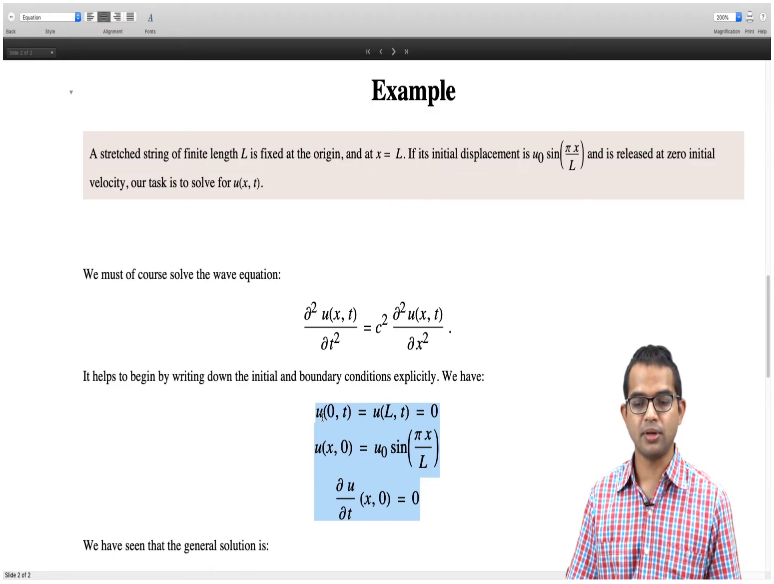

So, let us look at one example, which is the example of the stretched string, right. So, where in fact, these two kinds of waves conspire in a very special way, right. So, we are given this stretched string and it has length L, one end is fixed at the origin and the other end is fixed at x equal to L, right.

So, initial displacement is given to be this function and it is released at rest, let us say, for simplicity. And our task is to find we also take these very simple initial conditions, right. So, our task is to solve for u of x, t. So, it is possible to just start from this general formal solution and work out this problem involving these particular boundary conditions.

So, yeah. So, this is the differential equation. The boundary conditions, it's good always to write down explicitly the boundary conditions in terms of the equations. So, u of 0 comma t, 0, u of L comma t is 0, u of x comma at time t equal to 0, it is just this function. And dou u by dou t at time t equal to 0 is just 0, throughout, right. So, there is no velocity for the wave at time t equal to 0.

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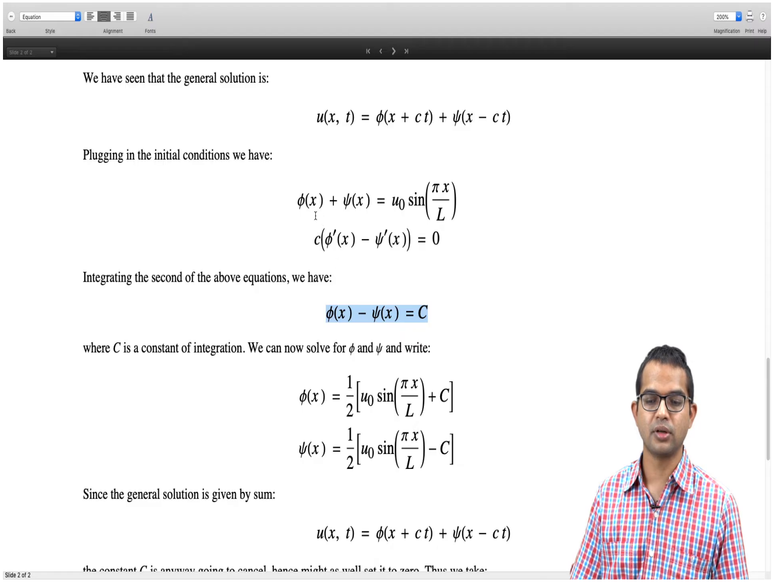

Now, this is the general solution. So, all we have to do is fit this general solution along with these boundary conditions and it is quite straightforward to do. So, if you plug in the initial conditions, then we have phi of x plus psi of x is equal to this function and then phi prime.

So, if you take the derivative with respect to time and then put t equal to 0, so you get this equation. So, basically, the derivative of the derivatives of these functions are the same, which is the same as saying that phi of x and psi of x are different from each other at best by some constant c, right.

So, if once we have this, we can go back and write down phi of x and psi of x according to this. So, you get phi of x is equal to u naught sin pi x by L plus c, the whole thing divided by 2 and psi of x is equal to half of u naught sin of pi x by L minus c, right. So, the general solution is I mean after all we are interested in you know we have to put back the time. So, this is you know at t equal to 0. We have worked this out, and then the structure of these functions have to be like this because of the initial conditions.

So, the general solution is given by this. So, you see that when you plug in this form, and this form, you know because you have this sum, this c is anyway going to cancel. In the final solution of interest for us, the c is actually not going to play any role. So, for simplicity, we might as well just set this constant to be 0, right. So, we will do that.

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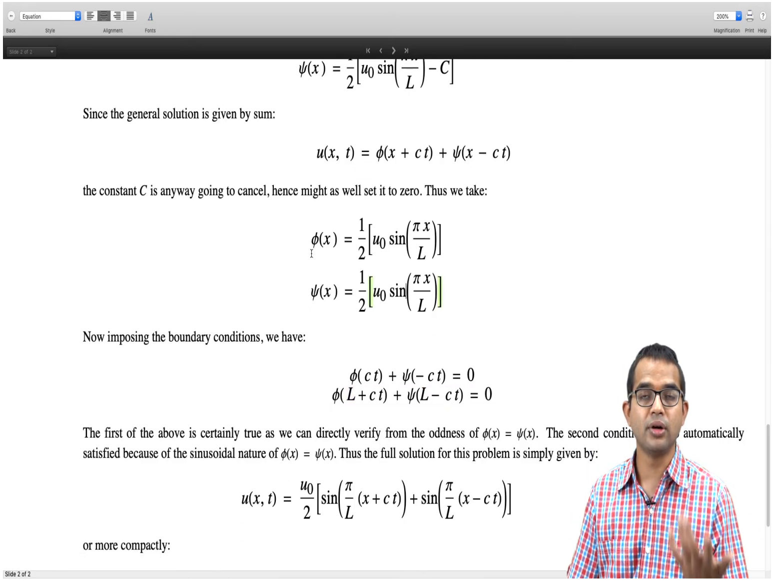

And so, then we will just work with phi of x is equal to this and psi of x is equal to this. So, in other words phi of x is actually the same as psi of x. Both these functions are the same, and now it is straightforward to impose the boundary conditions which are given here, right. So, at x equal to 0 and x equal to L, they have to be pinned, right.

So, then you have phi of ct plus psi of minus ct equal to 0. And, but phi and psi are really the same, so basically, you get this to be an odd function, right. And which we already have actually sort of inbuilt into this. So, we considered a very simple initial condition, such that the problem became actually straight forward.

And the other condition is phi of L plus ct plus psi of L minus ct equal to 0, and we have, our function is odd. So, indeed this holds all automatically as you can verify. And the second condition is also automatically satisfied because this sin function goes to 0 at x equal L, right. So, you can check this I mean; so, you have you if you put you know in the place of this argument, you have to just put this, and in place of this argument you have to put this.

And then you know when you have L, so you get pi plus and pi minus in other cases you can check from the property of sinusoidal function. That indeed it is automatically satisfied. We do not have to do any extra work. So, this is because we have chosen an extremely simple initial condition, right.

So, the full solution of the problem is then simply given by the sum of this, right. After all, we have to do this. And it is straightforward to write it like this. And in fact, there is a more compact way of writing this which comes from just some basic trigonometric identity adding two sines, you can rewrite this as u naught sin pi x by L times cosine pi ct by L. You can check this, ok.

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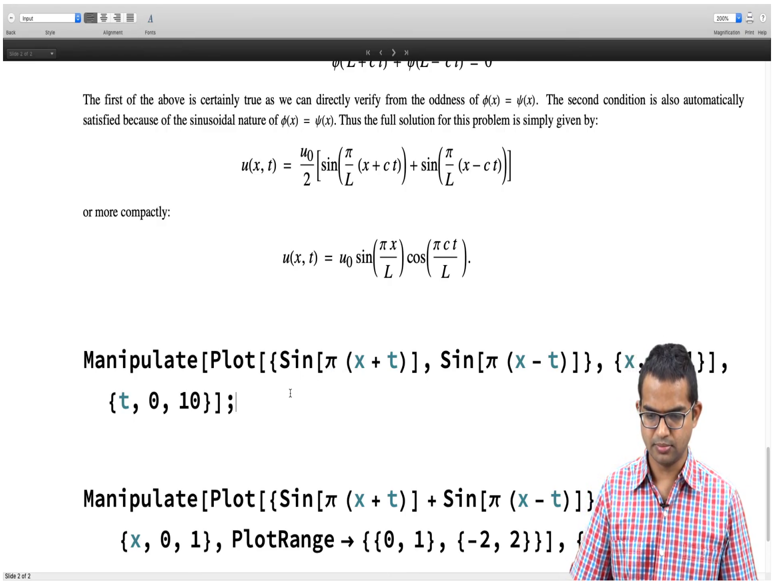

So, it is instructive to plot this function. So, let us first plot the sum or separately each of the terms, and then we will plot this sum.

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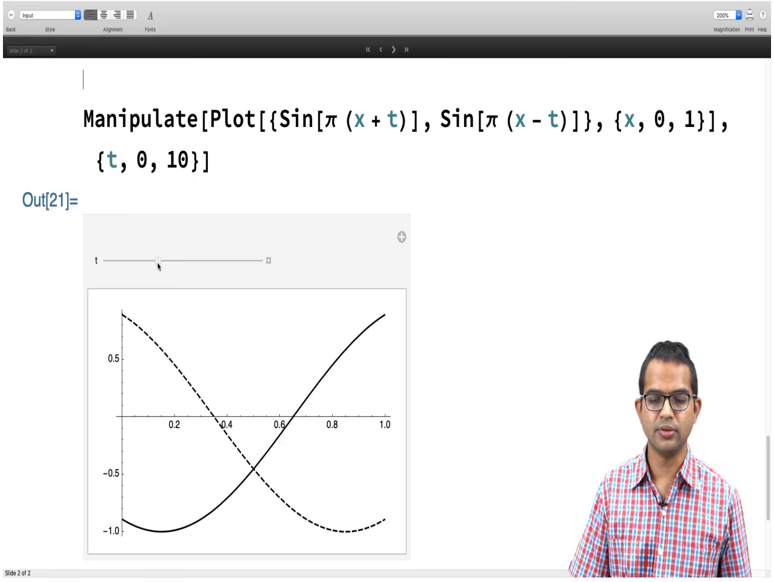

So, here I am plotting you know; there are these two composite guys. I mean of course, for simplicity I am taking c to be 1 and L to be 1, you know is just I mean as far as you do not even need u naught at this point. I am plotting this and this separately, so you can see that I mean it is time which is going to vary as a function of time. The shape of this curve is going to vary.

So, you see that there is a right moving wave and then there is a left moving wave, right. So, the two components are the two waves, one of them moves to the right and the other one moves to the left. And so, there is this periodicity and there is a periodic time interval, at which both of them are going to just merge with each other, right, ok. So, both of them are traveling waves. But together they can conspire to give us you know this what is called a standing wave.

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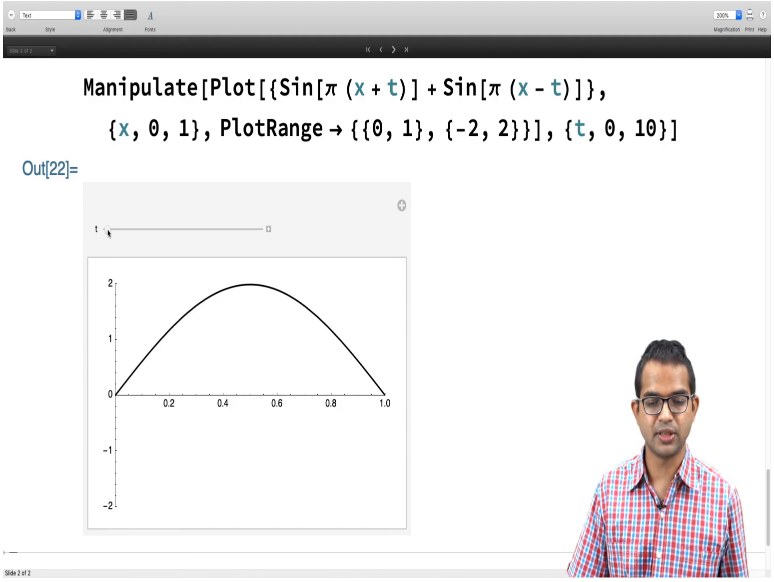

So, here I have added the two. Now, you see that as I increase the time, this is the initial time condition which has been chosen very, in a very special way here.

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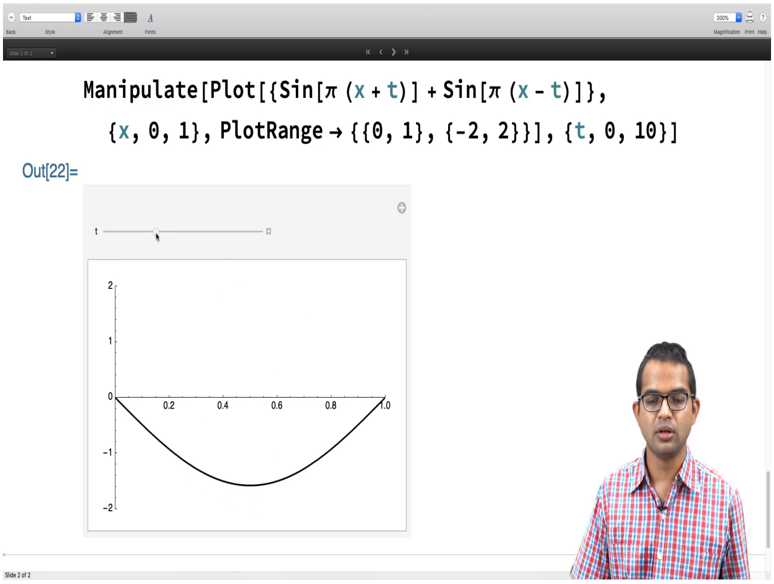

So, you see, this is what you get, ok. There are two traveling waves which can conspire to give us a standing wave, right, which is more appropriate for further description of a stretched string, ok.

Thank you.