Mathematical Methods 2

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Complex Variables
Lecture - 08
Cauchy-Riemann Equations

So, we have seen how the idea of a derivative is somewhat involved when we are working with a function of a complex variable and we have seen how differentiability is a rather demanding condition upon a function. So, in this lecture we will look at what kind of properties a function which is differentiable at a certain point, what kind of conditions must it satisfy and which will allow us to derive the so called Cauchy-Riemann conditions.

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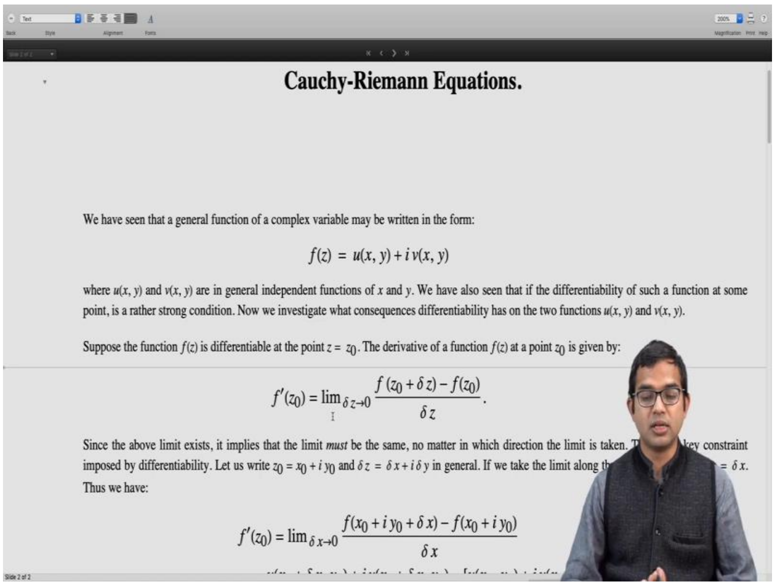

So, we start with this general form for a function of a complex variable. So, we have f of z is equal to u of x,y plus i times v of x,y, where both u and v are in general independent functions of x,y right. So, it turns out that you cannot have any arbitrary functions u and v, if you want this function to be differentiable at some point even if u and v are extremely nice functions in functions of two variables.

So, you might think that you know basically the calculus of functions of a complex variable is really the calculus of functions of two variables, but it is not just that. So, this u of x,y and v of x,y cannot be anything you wish. So, let us see what conditions are satisfied by a function which has a derivative at some point z0 and what conditions must these u and v obey for such a function.

So, let us start with the definition of derivatives. So, f prime of z0 is the limit delta z going to 0 f of z0 plus delta z minus f of z0 the whole thing divided by delta z. Now, well this limit exists, so, which means that it must be the same no matter in which direction you approach this point z0 right. So, that is the core of the condition. So, since you have all these infinitely many different directions.

So, demanding that this limit is the same in all these different directions will force the real part and the imaginary part of this function to obey some conditions and these will be called the Cauchy-Riemann conditions. We will work them out.

So, let us see what happens if you take this limit - the limit is defined in a general way at this point. Suppose we take this limit along the real axis. So, then we would have delta z is equal to delta x. So, if we do this then we have f prime of z is equal to limit delta x going to 0 and in place of z0 we put x0 plus i times y0 plus delta x because delta z is delta x minus f of z0 is the same as f of x0 plus i times y0 and then the denominator we have in place of delta z we write delta x.

So, but we know that this function f of this whole stuff is the same as u of x0 plus delta x you have the same argument x0 plus delta x. So, that is the real part and then comma y0 plus i times v of you know the real part x0 plus delta x comma y0 minus I am just expanding out this function and writing it in terms of u's and v's.

So, I have u of x0, y0 plus i times v of x0, y0. So, the whole thing has to be divided by delta x.

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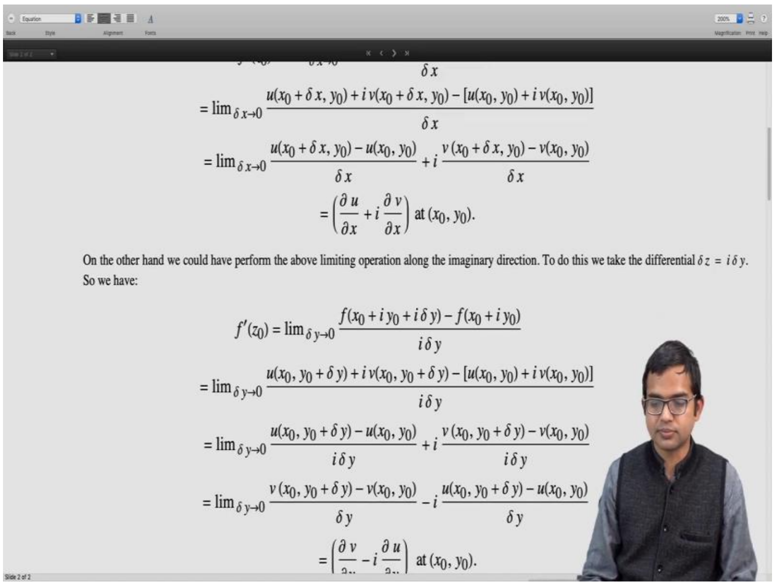

And so, this is equal to u of x0 plus delta x, y0 minus u of x0,y0 the whole thing divided by delta x plus i times v of x0 plus delta x, y0 minus v of x0,y0 divided by delta x. So, now, this is nothing but the partial derivative of u and of v well there is an i sitting here as well because when you are taking this limit delta x you are making a slight perturbation only along the x direction right.

So, you have dou u by dou x and in place of this is the second term is nothing, but i times dou v by dou x is the partial derivative of this function v at x0, y0. Now, let us see what would have happened if we had performed the same type of limiting operation, but from a different direction.

Let us take the y direction suppose we take delta z to be i times delta y. So, there is no delta x. So, it is just purely i times delta y. So, then we would have invoking the same definition. So, f prime of z0 is equal to limit delta y going to 0, right. Its i times delta y going to 0 which is the same as delta y going to 0, f of now in place of z0 plus delta z I will put x0 plus i times y0 plus i times delta y right.

So, it is like z0 is of course, just x0 plus i times y0, but delta z is i times delta y. Then of course, I have to subtract f of x0 plus i y0. Now, the whole thing has to be divided by i times delta y, because that is the small differential element and then we write this as limit delta y going to 0 u of x0, y0 plus delta y right in place of the function f, I am writing it as the real part and the imaginary part separately.

So, u of x0,y0 plus delta y plus i times v of x0,y0 plus delta y. So, then we have to subtract out this function f of x0 plus i y0 which I will also write it out explicitly as u of x0, y0 plus i times v of x0,y0. The whole thing of course has to be divided throughout by i times delta y. So, then again I separate out these two terms and write it in a suggestive form.

So, I have limit delta y going to 0 u of x0,y0 plus delta y minus u of x0,y0 divided by i times delta y plus i times v of x0, y0 plus delta y minus v of x0,y0 divided by i delta y. Now, well I mean it is convenient to put the second term first and the first term second.

So, now, I have v of x0,y0 plus delta so, the i has cancelled and so, we see that the first term is actually nothing but a small perturbation of delta y, you are asking what happens to the function v if you change delta y by with the y coordinate by a small amount.

So, the difference divided by delta y. So, it is like a partial derivative and again u also has the second the first term has become the second term and so, we have a minus u of x0,y0. So, this should be a minus i times. So, there is an i sitting here. So, I need when I write the next step, so, there is the i here.

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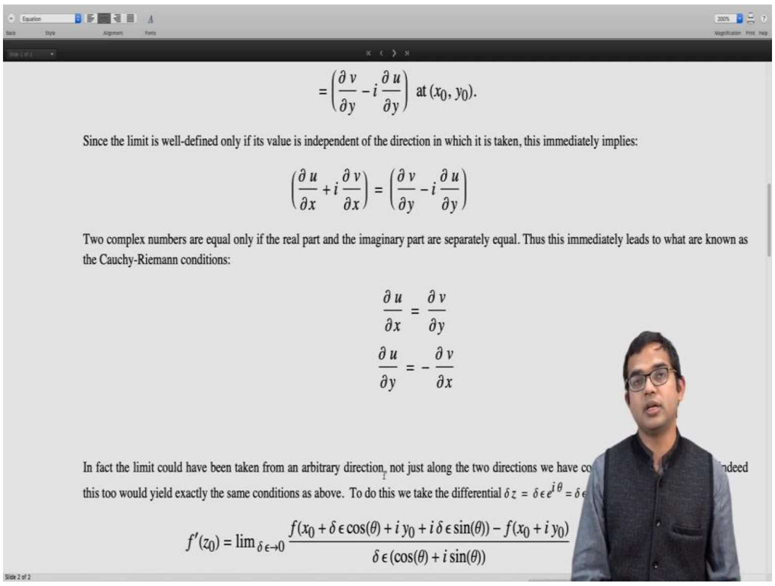

So, its dou v by dou y minus i times dou u by dou y ok, all of this has to be evaluated at the point x0,y0. So, the key point is that if you work out this derivative according to first principles by taking the limit along the x axis along the real axis you get one answer and if you work out the same derivative you know by taking the limit along the imaginary axis you got a different answer right.

So, the only way to make sense of these two different answers is to demand that both these answers have to be the same. So, your u and v have to be such that these two answers must be the same and so that is what leads to the Cauchy-Riemann condition.

So, immediately it implies that dou u by dou x plus i times dou v by dou x must be equal to dou v by dou y minus i times dou u by dou y right. When two complex numbers are equal the only way that can happen is if the real parts and imaginary parts are both separately equal. So, which means dou u by dou x must be equal to dou v by dou y and dou u by dou y must be equal to minus dou v by dou x.

So, that is these two Cauchy-Riemann conditions, whenever a function is differentiable at some point then necessarily it must satisfy the Cauchy-Riemann conditions. So, this is something that we could have actually obtained by taking the limit from an arbitrary direction.

It is a little more involved a little more messy in some sense but let us look at what would have happened if we had taken some other direction other than only x and y right. You should expect the answer to be the same, regardless of which direction you took it from. So, let us verify that this two holds. Suppose, we take this delta z to be delta epsilon times e to the i theta right. So, then we expand it out we have delta epsilon times cos theta plus i times sin theta.

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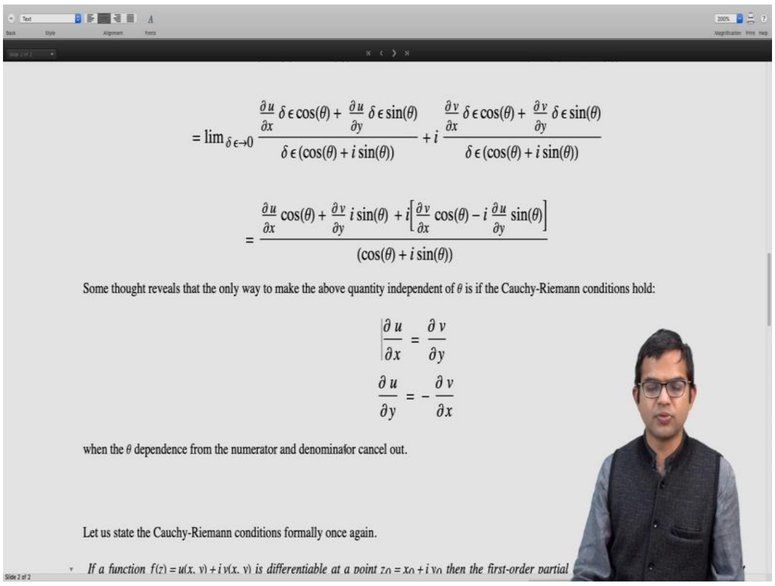

Then if you take the derivative f prime of z0 then you have limit delta epsilon going to 0 f of you know this more complicated argument, now x0 plus delta epsilon cos theta plus i y0 plus i times delta epsilon sin theta. So, this just comes from a more complicated delta z minus of course, just f of z0 which is f of x0 plus i y0 and in the denominator once again you have delta epsilon times e to the i theta. So, I have expanded out e to the i theta in the denominator.

Now, we can separate these terms out right conveniently. So, first of all we will write f in terms of u and v and so, I have and then I will collect terms. So, that u's appear on the first term and then v's appear in the second term right.

So, I have merged a few steps maybe a couple of steps into one and so I have u of x0 plus delta epsilon cos theta, y0 plus delta epsilon sin theta minus u of x0,y0 divided by this differential element and then plus i times v of x0.

So, there is a perturbation both in the x direction and in the y direction. So, you have to consider v of x0 plus delta epsilon cos theta, y0 plus delta epsilon sin theta minus v of x0, y0 divided by delta epsilon you know times e to the i theta.

So, now, we argue that. So, this is nothing but the small perturbation of you know the arguments slightly away from x0 and slightly away from y0 which is the same as saying dou u by dou x times dou epsilon cos theta plus dou u by dou y delta epsilon sin theta and then there will be some higher order terms which anyway will go to 0 because we are going to be taking this limit delta epsilon. So, we will just keep the terms up to first order and u of x0, y0 anyway will cancel.

So, I mean I am just expanding it is like a Taylor expansion of a function of a real variable right u and v are functions of real variables. So, this is the usual kind of Taylor series expansion. So, indeed I get only two terms here and likewise this term which comes along with this i also needs only two terms because this v of x0,y0 will cancel.

And then the first order terms are all up keeping higher order terms in any case will go to 0, because you have only delta epsilon sitting in the denominator delta epsilon squared of course, will just give you 0. So, first order terms are all I have to keep and delta epsilon cancels through all.

So, I collect terms in a suitable way here I get dou u by dou x times cosine theta plus dou v by dou y times i sine theta. I have collected terms in a suggestive way plus i times dou v by dou x cos theta minus i times dou u by dou y sin theta the whole thing has to be divided by cos theta plus i sin theta.

So, what do we expect this quantity to be? We expect that this quantity to be independent of cos theta plus i sin theta right. So, maybe there is a way to argue for this more rigorously and maybe you can take like a derivative with respect to theta and show that you know the only way they will be this quantity will be independent of theta right this is just a this is a function of a single variable theta.

So, definitely you can argue that if it has to be independent of theta then its derivative with respect to theta must be 0, right. So, but you can also just directly look at this equation and immediately convince yourself that the only way that this entire thing can become independent of theta is if you can pull out a factor of cos theta plus i sin theta from each of these terms.

So, the first term you will get you can pull out a factor of cos theta plus i sin theta provided dou u by dou x is equal to dou v by dou y. So, that is the only way this is going to happen and likewise if dou v by dou x is equal to minus dou u by dou y you can pull it out and then you will get a; you will get a factor of cos theta plus i sin theta. So, you need another plus sign here. So, you need dou v by dou x to be equal to minus dou v dou u by dou y which is nothing but the Cauchy-Riemann conditions right.

So, it is possible to argue in general for considering an arbitrary direction, but it is easiest if you are working along the x direction and the y direction that gives you the Cauchy-Riemann conditions in this you know in the standard Cartesian form right. So, these are the Cauchy-Riemann conditions.

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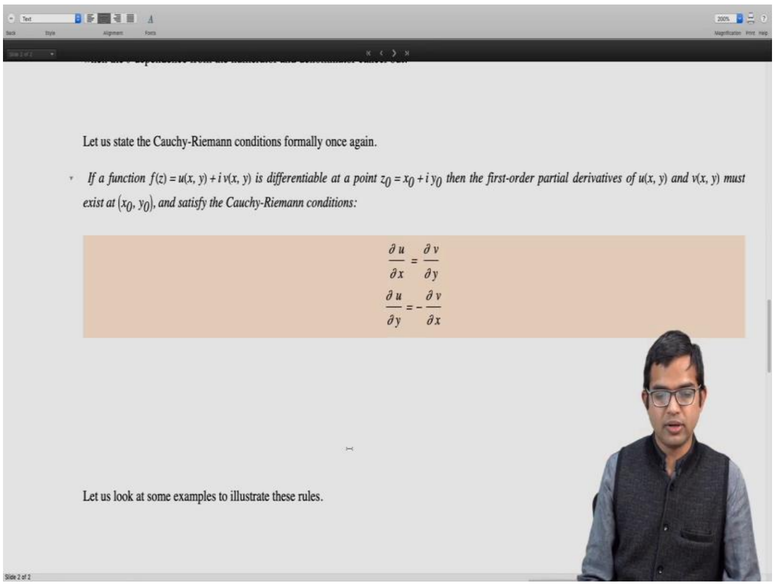

Let us look at some examples of how this plays out. So, but before we do that let us state the Cauchy-Riemann conditions a little more formally again right. So, whenever a function f of z is differentiable at some point z naught then well Cauchy-Riemann conditions hold which immediately which presupposes that these partial derivatives of u and v must exist at that point x0,y0 and they are not only exist, but they also satisfy the Cauchy-Riemann conditions.

So, you cannot just take two arbitrary functions u of x,y and v of x,y and expect to just make one of them the imaginary part the other the real part and get a nice overall function of a complex variable, you have to pick them up in such a way that dou u by dou x must be equal to dou v by dou y and dou u by dou y must be equal to minus dou v by dou x.

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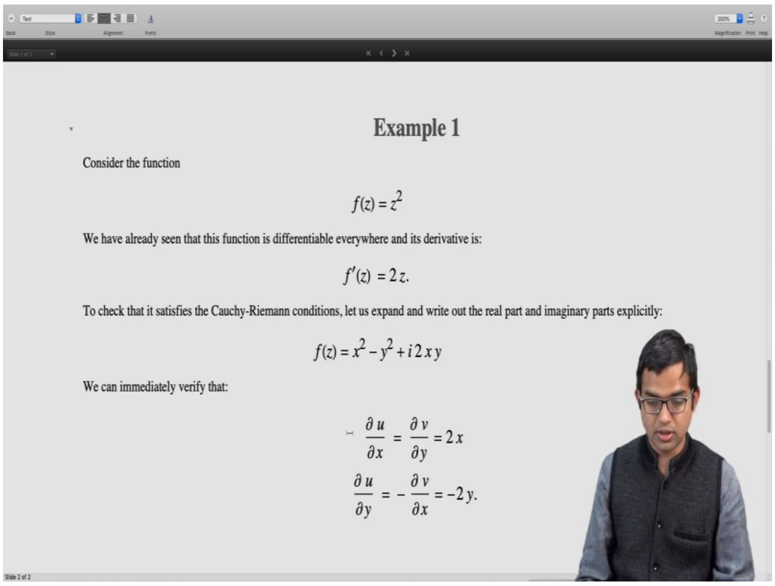

So, let us look at a few examples to illustrate these ideas. So, let us start with our favorite function f of z equal to z squared, we already seen that this is differentiable everywhere and its derivative is simply given by 2 z. We already know this. So, let us check the Cauchy-Riemann conditions for this function. So, the real part and the imaginary parts for this function also we know u of x,y is just x squared minus y squared and v of x,y is 2 xy.

So, dou u by dou x is definitely equal to dou v by dou y because both of them are equal to x here and likewise dou u by dou y is just minus 2 y and so, is minus dou v by dou x. So, minus 2 y so, indeed both the Cauchy-Riemann conditions hold ok. So, not a surprise this is a function whose derivative is well defined and therefore, Cauchy-Riemann conditions hold at all points.

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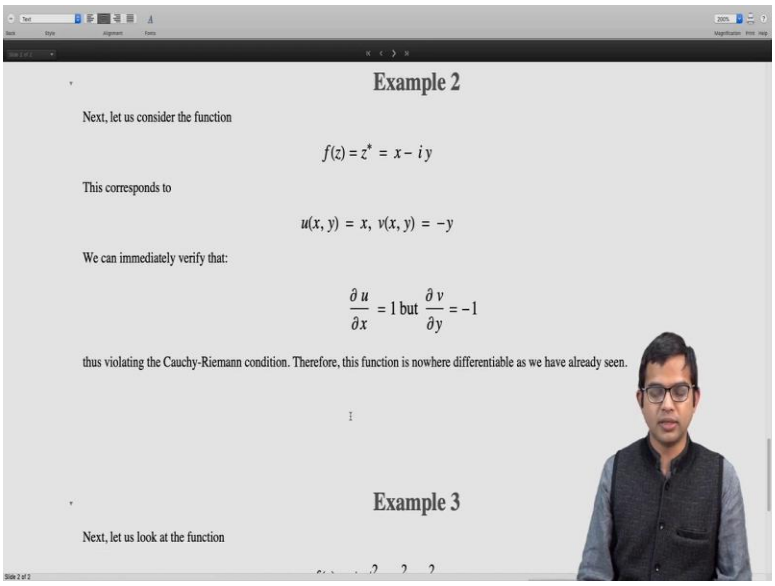

So, it is important to emphasize that the Cauchy-Riemann conditions are necessary conditions for differentiability right. So, what we have shown is that if a function is differentiable from first principles then for sure Cauchy-Riemann conditions hold right. So, they are not quite the sufficient conditions, but you know they are almost, but not quite.

So, we will come back to that point a little bit later, but because there are necessary conditions for differentiability what it implies is if a function does not satisfy Cauchy-Riemann conditions at any point then for sure it is not differentiable right. So, this is you know if a implies b then not b will imply not a right.

So, if a function is differentiable therefore, Cauchy-Riemann conditions hold that is the same as saying if the Cauchy-Riemann conditions do not hold then it is not differentiable right. So, let us look at an example of how this plays out. Suppose we consider this function f of z equal to z star. We already looked at this and we saw how this function is not differentiable right.

So, it is something that we could have you know we could work out using Cauchy-Riemann conditions, because u of x,y here is equal to x and v of x,y equal to minus y and immediately we can see that dou u by dou x is equal to 1, but dou v by dou y is minus 1. So, there is no way that you know these two can be equal and there is no point x,y for which these two will be equal.

So, therefore, Cauchy-Riemann conditions are not satisfied at any point and therefore, indeed this function f of z is equal to z star is not differentiable right. So, this is a conclusion that we already arrived at using first principles using the definition of the derivative.

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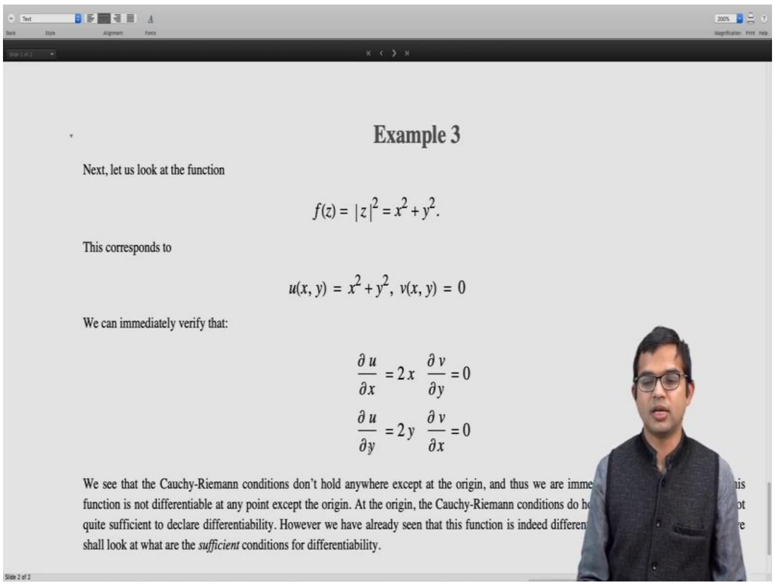

Let us look at one more example where we had difficulties when we were trying to take the derivative which is the mod of this z squared function which is just x squared plus y squared. So, this corresponds to u of x,y is equal to x squared plus y squared and the imaginary part is 0 right. So, immediately we can see that dou u by dou x is dou x and dou v by dou y is 0 dou u by dou y is 2 y and dou v by dou x is 0 because v itself is 0.

So, we see that basically the Cauchy-Riemann conditions do not seem to hold except at one point and that is the point where x equal to y equal to 0 which is the origin, if z is 0 then the Cauchy-Riemann conditions hold right. So, what we can say for sure is when the Cauchy-Riemann conditions do not hold, which is at every point other than the origin for sure this function is not differentiable right this is something that we already know.

We have already come to this conclusion from first principles, but the fact that the Cauchy-Riemann conditions do hold at the origin is not is does not immediately guarantee that it is going to be differentiable, but we have already seen that it is differentiable at the origin and therefore, it is indeed we do expect that Cauchy-Riemann conditions will hold as they do in this case right.

It turns out that the sufficient conditions for differentiability are a little more than Cauchy-Riemann conditions. So, this is a somewhat of a technical and a subtle point, but we will have a short discussion about this as well, but in this lecture we just want to point out that the necessary conditions that a function is differentiable is that it must satisfy Cauchy-Riemann conditions.

And this can often be used to check for Cauchy-Riemann conditions and see if it does not hold then for sure the function is not differentiable at a certain point, that is all for this lecture.

Thank you.