

Our Mathematical Senses

The Geometry Vision

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Lecture - 20

Video 4E: Incidence relations in the extended plane

So I want to take a closer look at the incidence relations in this extended Euclidean plane that we've just built. So to start with, let's just ask the question, is the following statement still true? The question is that any two distinct points are incident with exactly one common line. Given any two points, there's a unique line that they determine in this extended Euclidean plane P_2 . So does that continue to hold even though we added all these points at infinity? Well, yes it does. We saw this already when we were showing that P_2 is a linear space. We saw that if we take an ordinary point at a point at infinity, they will have a unique line between them.

And if we take two points at infinity, they will have a unique line between them, namely the line at infinity. So when we built P_2 as a linear space, we saw that this statement is true. What about the following statements? In fact, as a question, which statement best describes the corresponding analogous situation with two distinct lines? We could say that any two distinct lines are incident with at most one common point, any two distinct lines are incident with exactly one common point, or that any two distinct lines are incident with at least one common point. Back in the Euclidean plane, this was true.

Any two distinct lines could be parallel or they could intersect, so they'd have at most one point in common. So now what's the situation in the extended plane? Let's consider the cases. We could have two ordinary non-parallel lines. We could have just two lines that are ordinary in the ordinary plane, which are not parallel. They'll intersect at one point, as before.

We could also have two ordinary parallel lines. Earlier they didn't intersect, but now they're going to intersect at a point at infinity. This is L , this is M . We can call this either PL or we can call it PM . Those are the same thing, because they're in the same family.

So they will also intersect at exactly one point. What about an ordinary line and the line at infinity? Any ordinary line is going to intersect the line at infinity at exactly one point. Namely, if that line is called L , it'll intersect the line at infinity at the point PL . So again, one unique point of intersection. This is the most appropriate statement in P2.

So we now have two very similar statements about points and lines. Any two distinct points are incident with exactly one common line, and any two distinct lines are incident with exactly one common point. They're remarkably symmetric. If we exchange the words point and line, we literally transform one statement into the other. So this is what we call duality in P2.

We say that these statements are dual to each other. In the same way, we can actually dualize any statement in projective geometry. If we write the statement in terms of incidence, then we just switch the word point with the word line and the word line with the word point. We exchange those words and we get a different statement in projective geometry. But remarkably, just like both these statements are both true, any statement in projective geometry that's true, well, its dual will also be true.

So we can dualize any statement in projective geometry, any true statement, and get another true statement. So statements are true if and only if their duals are true. We haven't proved this, by the way. This is just something I'm telling you, which can be proved by developing projective geometry very carefully and axiomatically. And we're not going to prove it in this course, and we're not even going to use it, strictly speaking, but it's a nice thing to keep in mind.

As we encounter more statements in projective geometry, it can be very useful to consider what the dual statement is saying. So in this course, we're going to take a slightly more informal approach. Everything we say, do is still going to be rigorous and precise, but we're not going to always go back to the axioms. So we're not even going to give you the axioms precisely. But this is still a very useful thing to keep in mind, that given any statement, it comes along with a dual statement.