Our Mathematical Senses

The Geometry Vision

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Lecture - 21

Video 5A: Coincidence #1 the harmonic conjugate theorem

So today we're going to look at some of the content of projective geometry, some surprising results that involve just points, lines and intersections. And we're going to do that in the form of three coincidences. So remember that projective geometry is the study of points and lines in the extended plane P2. But you might think, what can we really say? We just have a bunch of lines which meet in points and a bunch of points which join to form lines. And it seems like we can only make statements of incidents, which is true. But thankfully, projective geometry is full of surprising, unexpected meetings of points and lines, also known as coincidences.

So here's an example of a coincidence. It's a straightforward construction. This here is basically a recipe, a construction, and it gives us a lot of freedom. There's a bunch of choices we can make, and we'll see that in a second.

But somehow, despite all that freedom of choice, in the end of the day, we get a surprising incidence relation, which holds no matter how we make those choices. So that's the part that feels like a coincidence, and indeed is a coincidence. So let's try this out. So let's just do the recipe. Let's just follow the recipe.

Let's just do the construction. So let A, B, and C be three points on a line L. This is our line L. A, B, and C are three points. They're collinear.

Let's choose a point P that's not on L, somewhere off of L. And now let's choose any point on Q on the line PB. So first let's draw PB, and then let's just choose a point Q anywhere on PB. So let's choose one here. So that's our second choice.

We could choose P anywhere we wanted. We can choose Q anywhere along that line. Now let's let R be the intersection of AP and CQ, and let's let S be the intersection of AQ and CP. It'll be a little easier if we draw these lines AP, AQ, CQ, and CP. Basically, we want to connect A to P, A to Q, C to P, and C to Q.

So we'll connect up C to P and Q first. Then we'll connect up A to P and Q. And then let's mark these points R and S. R is the intersection of AP and CQ. So this is R.

And S is the intersection of AQ and CP. So this is S. What's the next step? Let's let D be the intersection of RS with L. So we want to consider this diagonal. So diagonal meaning that we've created a quadrilateral here.

So there's one more diagonal, RS. And if we draw it, we can consider the intersection of it with our original line L. And let's call that D. So now the claim is this location of D depends only on A, B, and C. It's fully independent of the choices I made in choosing P and Q.

So you don't have to take my word for it. Let's see it in action. Let's choose a different Q. I'll keep my P here. But earlier I chose that Q.

Let's put Q higher up. Let's choose a Q there. And let's do the same construction. Let's connect CP, CQ, AP, and AQ. Let's mark those lines.

And let's mark R and S. Let's draw the diagonal between R and S. And boom, it actually hits that point again. Hits it square on, straight on.

That same point D. So we get the same D again. OK, let's try again with a different P. Rather than choosing a P here, let's choose a P over here. We mark the line PB. We mark Q somewhere along that line.

Again, that's a totally free choice we have. We connect CQ, CQ, P. We connect AQ and AQ. We mark R and S. We draw the line RS, the diagonal.

Here's our quadrilateral this time. Let's draw this other diagonal. And yep, it worked perfectly. It goes through that line D again. So in every situation, the location of D seems to depend, does indeed seem to depend on A, B, and C.

Of course, I've only given three examples. You can try it yourself and try other places of putting P and Q. And I encourage you to try it once just to see it in action from your own pencil. So now I want to make a definition, the harmonic conjugate. And basically, if we follow this construction, the harmonic conjugate, this point D that we constructed is known as the harmonic conjugate of B with respect to A and C.

And it's denoted HAC of B. So this point D is the harmonic conjugate of B with respect to A and C. So if you believe that this construction is true and this D is independent of all these choices and only depends on A, B, and C, then we might as well give it a name that makes it clear that it's dependent on A, B, and C. And that's the harmonic conjugate. So we can actually rephrase this coincidence as the harmonic conjugate theorem. And in that case, what the theorem states is if you follow this construction, HAC of B does not depend on the choices of points P and Q.