Our Mathematical Senses

The Geometry Vision

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Lecture - 22

Video 5B: Proof of the harmonic conjugate theorem

Alright, so how do we prove the harmonic conjugate theorem? The collection of points and lines that we've drawn here, it actually might look a little familiar. In fact, it looks a lot like the perspective views of the tiled floors that we've been investigating. These kind of look like vanishing points of the sides and diagonals of this quadrilateral. So can we interpret this configuration as a perspective view of a square tile, a single square tile, namely this tile here? Could this be a perspective view of a square? Well, it's hard to say. We haven't really investigated or figured out whether any old quadrilateral could be the perspective image of a square. A square is kind of special.

And we'll look into that soon, but for now let's not get into that. Let's not make any assumptions that are not so easy to back up. Instead, let's be a little more conservative. We can interpret this quadrilateral as the perspective view, as the perspective image of a parallelogram.

That we can do. We can say that, okay, this is a perspective drawing, and in space, on the floor plane, this is a parallelogram. Its sides converge to this vanishing point, its other sides, opposite sides, converge to this vanishing point. So since these sides are parallel, and these sides are parallel, in the top-down view, it's a parallelogram. So let's do that.

We can interpret it as the perspective view of a parallelogram tile. And let's see what our construction that we just did, how does that construction play out from a top-down bird's eye view, where this will actually look like a parallelogram. So here's a bird's eye view. Here's the points Q and P here. Oh, I didn't mark R and S here, just so it's less cluttered, but we have R and S over here.

And let's just perform the construction again. So we have this. To prove the harmonic conjugate theorem, we have to show that if we do the construction again with different Ps and different Qs, with a different Q, we'll still get the same D, the same harmonic conjugate point. So let's do the construction again with different choices of P and Q, but keep track of everything we're doing from this bird's eye view. So here's another point P'.

Here's another point Q'. We connect C up to those, and we connect A up to those to get our new quadrilateral. And now we have our new points R' and S', and look at their diagonal. And if the theorem is true, that diagonal should intersect the same point D. And it looks like it does, but we have to prove that it does.

So we're not going to assume that yet. So that's what we have to show, that this diagonal line here between R' and S', that ought to go to the same point D. In other words, in the top-down view, this ought to be parallel to this line here. This diagonal and this diagonal ought to be parallel. And if we can show that, then we're done.

So let's see how this second tile will look in the bird's eye view. Well, here it is. It's another parallelogram. Its vertices are Q' and P', R' and S'. And its edges, its side edges are going to be parallel to the side edges of the first parallelogram, because they share that vanishing point.

This top and bottom edge are going to be parallel to these edges, because they share this vanishing point A. And even its green diagonal is going to be parallel to this green diagonal, because they both share this vanishing point B. So in particular, we have that R', P' is parallel to Rp, Q', S' is parallel to Qs, R', Q' is parallel to Rq, P' S' is parallel to Ps, and Pq is parallel to P' Q'. Again, what we have to prove is that the diagonal R' S' is parallel to the diagonal Rs. So let's just focus on the top-down view, the bird's eye view, and see if we can prove this.

Given all of these sets of parallel relations, can we conclude that R' S' is parallel to Rs? If you're familiar with vectors, vector calculus, or just even the algebra of vectors, then you might want to take a moment to pause and try this on your own. It's a nice exercise. So pause and try it on your own if you want, but I'll also do it now to help you get a sense of how it can be done. So the key is to define vectors u, which is equal to Rp. Oops, let me use the right color.

So we define a vector u here, which is just Rp, and let's define a vector v, which is equal to Rq. So I have my vectors u and v. And now let's define, given those vectors, we have that u plus v is what? Well u plus v is just this vector, this diagonal vector Rs. Or rather it's this diagonal segment Rs. And u minus v is just the vector running from q to p.

This is u minus v, and this is u plus v. So the side vectors and the diagonal vectors are all nicely describable via u and v. Now what else do we know? Because of all these parallel relations, we know that the triangle Rqp must be similar to the triangle R'q'p', because all of their corresponding edges are parallel. Because they're similar, it follows that u is the vector Rp. There has to be some scalar multiple c such that R'p' is just c times u.

And R'q' is just c times v. So there's some scalar c that relates these two similar triangles and gives us these relations. So it follows that R's', this diagonal in question, well that is equal to cu plus cv. But cu plus cv is just c times u plus v. Just factoring out the c.

So u plus v is Rs. So R's' is just c times Rs. So R's' is just parallel to Rs. As a vector, it's just a scalar multiple of this vector. So they are going to be parallel. Therefore the line R's' will intersect the line AC, our original line AC, at that same point D.

It's going to intersect it there. Because this line and this line are parallel. They'll share in this perspective view, they're going to share a vanishing point. Hence the harmonic conjugate D, which we denote HAc of B, is indeed well defined. So what have we proved here? Well, we've proved the harmonic conjugate theorem.

We've proved that given any three collinear points, A, B and C, the harmonic conjugate HAc of B, which is defined via this construction that you see here, pick B and Q, draw this quadrilateral, draw the other diagonal, boom, you get your point. That's how we define our harmonic conjugate point. Well that's well defined. The construction does not depend on the choices of P and Q. So that completes the proof.

But if you're a bit more cautious than I've been, you might wonder are there any gaps in our proof? In other words, have we really chosen fully arbitrary points P and Q? Or have we been a little careful so that our points P and Q allow us to work things out nicely? And I've cheated a little bit. I have cheated a little bit. For example, I always chose P and Q to lie on the same side of this line, L. If I'd chosen a point P here, drawn PB, and drawn Q over here, for example, well suddenly it's a bit harder to see how we get our quadrilateral. I connect up A to P and Q, I connect up C to P and Q, and I get this stuff here, but that's a quadrilateral, but it's not a perspective view, doesn't seem to be a perspective view of a parallelogram at a first glance.

So in a way we have to be more careful, but we'll see in the next chapter that we don't have to be much more careful. We can do a small tweak and this interpretation actually will work even in a weird situation like that. So we'll keep that in the back of our mind, but don't worry too much. Now a final thing I want to take note of though is that as we vary the points A, B, and C along a line, we fix a line L and vary the points A, B, and C, the harmonic conjugate will move around. So here is a screenshot from the software GeoGebra.

A, B, and C are these three points and you can see the construction being carried out. B and Q are here, we get our quadrilateral, we draw the diagonal RS, and we get our D. But now let's move the point B a little bit. Here B is kind of closer to C, but let's move it a little closer still. So as we move B closer to C, what's happening to D? It's also moving closer to C.

So we can, as we shift B, D also gets shifted. Let's see another example. Here B is a little closer to A, and D has actually moved over to this side, because with this B, if we chose a P here, connect it to B, draw our quadrilateral, suddenly now our diagonal is pointing this direction and hitting the line L, our original line L over here. So suddenly D is over here. What if we move B a little closer? D moves closer as well to A.

So you can see that as we move D, as we move B, D also shifts. Let's look at one final example where B is exactly in between A and C. A, B, and C are evenly spaced. So what happens now? Well, if we pick a point here, connect it to B, pick our Q, draw our quadrilateral, our diagonal R, S is not intersecting our original line at all.

It's parallel. There is no D that we can see. So in this case, R, S is parallel to our original line AC, or L. So HAC of B will still exist, but it's going to be a point at infinity. It's going to be the point at infinity, well, I didn't mark L, really I should say PL, or I can say PAC. L is just AC here, I should have written that.

So this is our point L. So our harmonic conjugate is going to exist on this line, but it's going to exist at infinity. That's where this line and this line meet. Somewhere over there. So the takeaway is that the harmonic conjugate is only defined in P2, not in R2. Because if we were just working in R2, it would simply not be defined in this case.

So we better revise our definition. Let's start by letting A, B and C be three collinear points in P2. Now we just go through the exact same construction, and now the harmonic conjugate of B with respect to A and C really is well defined. Final question though, what if either A or C lies at infinity? We've seen that the harmonic conjugate can lie at infinity, but now that we're working in P2, our original points A, B and C, what if some of them lie at infinity? So what if one of these lies, or what if both of them lie at infinity, A, B and C? In that case, the line at infinity will contain A, B and C, all of them. So what is the theorem saying in that case, where A, B and C are all on the line at infinity? I'll let you think about it a little, but actually we've already examined that case. In fact, we reduced the other cases to this case in order to prove the harmonic conjugate theorem.

Think about it a little, what this will look like when A, B and C are all on the line at infinity, but basically it means that the quadrilateral we draw will actually look like a parallelogram. So that was that bird's eye view case that we used to prove the general theorem.