Our Mathematical Senses

The Geometry Vision

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Lecture - 23

Video 5C: Coincidence #2 Pappus's theorem

We're now going to review a second coincidence, Pappus's theorem. So given one set of collinear points, a, b, and c, and another set of collinear points, a, b, and c, the intersection points of the line pairs ab and ab, ac and ac, bc and bc, are all going to be collinear. So if we intersect ab and ab, that's x, we'll call that x. ac and ac, we call that y. bcc and bcc, we call that z. Those are going to be collinear.

And we'll call this the Pappus line. So this theorem holds for any two lines and any choices of points. So let's just see a couple more examples. I'm going a bit fast because you saw this already in the intro video, but we'll still go through a few more examples so you can see the variety of ways this can work out.

So we have two lines with capital A, capital B, capital C, and let's reorder the points here. So it's little c, little a, little b. The theorem will still hold. Little ab with a little b, that's our x. a little c with c little a, that gives us our second intersection point.

And finally, little cb with little bc, that gives us our third one. And this, this, and this point are indeed collinear. So we see the Pappus line again. Or let's do another example where the lines are crossing and the points are on different, on both sides. So we have abc and little a, little b, little c on this line.

So we get that little ab and little ba meet here. We get that little ac and little ca meet here. And finally, little cb and little bc meet here. So this point, this point, and this point are indeed collinear again. And that's where the Pappus line is in this third case.

Now let's do one more example which is maybe a little more unusual or a little more interesting. Let's assume these two point lines appear parallel. And let's label our points capital A, capital B, capital C, little a, little b, little c. What does Pappus's theorem look like now? So let's connect up little ab with capital A, little b. OK, these are also going to look parallel because of this configuration.

Similarly, little ac and little ca appear parallel. And little bc, little cb appear parallel. So where are x, y, and z? Where are the pairwise intersections? And where is the Pappus line? Is there a Pappus line? Are these points collinear? I mean, we have to first find these points. So it's important to remember here that Pappus's theorem is actually a statement about points and lines in the extended Euclidean plane P2. So in fact, little ab and little ba do intersect, but at a point at infinity.

These guys also intersect at a point at infinity, and so does this third pair. So all three points of intersection are points at infinity lying on the line at infinity. So it does still hold, but in this case, the Pappus line, well, let's actually see it. Let's take a perspective view of this image, tilt it down, and extend our lines. So now we can see where they intersect.

And indeed, all of these three pairs intersect at points at infinity, which lie on the line at infinity. So the line at infinity is the Pappus line in this case. And the theorem does still hold, but only because we're thinking of it as a statement about points and lines in the extended Euclidean plane. If we were just restricting our attention to the ordinary Euclidean plane R2, the statement would not hold, it wouldn't be true for this configuration. So as an exercise, I want you to think about a different situation where just one of the points x, y, and z lies on this line at infinity.

In other words, you can draw a configuration where just one of the pairs, say a little ab and little ba, appear parallel in this bird's eye view and intersect at infinity. The other pairs will... can you draw a configuration where the other pairs intersect on the plane? So if you can concoct a situation like that, and in that situation, what is Pappus's theorem going to say? What will its content be?