Our Mathematical Senses

The Geometry Vision

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Lecture- 25

Video 6A: The exended euclidean space P3

Our task in this lecture is to prove Desargues' Theorem, but in order to do that, there's some other business we have to take care of first. It seems hard to believe, but although Desargues' Theorem takes place entirely in the plane, the extended plane P2, is a statement about points and lines in the plane, and if certain conditions are met, it implies that some other things hold about those points and lines in the plane. It entirely lives in the extended plane P2. But in order to prove it, we're actually going to have to transcend P2, transcend this extended plane, and enter a larger three-dimensional space. But before we define this larger three-dimensional space, let's return to some perspective drawing. So I want to take a closer look at drawings of lines in space.

So far, we've associated points at infinity to various families of parallel lines on the ground plane, really to all the different families of parallel lines on the ground plane. And we've seen that in a perspective drawing, each family converges to a vanishing point, to its own vanishing point on the horizon line. So this family converges to this vanishing point, this diagonal family on the ground plane converges to this vanishing point, and this diagonal family on the ground plane converges to this vanishing point. And all of these vanishing points lie on the horizon line.

In fact, even lines off of the ground plane, if they're parallel to the ground plane, will appear to converge to those same vanishing points on the horizon line. So what do I mean by this? Well, look at this edge here of this roof. It is parallel to these diagonal lines here in this family. And it too will converge to that vanishing point on the horizon line. Or take this rooftop, take this ledge here, it's parallel to this family of diagonal lines, and it'll converge to their vanishing point.

So even lines off of the ground plane, if they're parallel to the ground plane, will appear to converge to those same horizon line vanishing points. What about lines that are not parallel to the ground plane though? What will happen to those? How will they appear? What will their images in the picture plane look like? So we've actually seen that by the vanishing point theorem, they ought to converge to some vanishing point in the picture plane. Because in the vanishing point theorem, we proved that any family of parallel lines, as long as it's not parallel to the picture plane, will converge to some point on the picture plane. There was nothing about being parallel to the ground plane. On the other hand, in the horizon line theorem, we stated that if the line is not parallel to the ground plane, then it won't lie on the horizon line.

But it should still converge to some vanishing point somewhere. So wait a second, so there's vanishing points off of the horizon line? Well, yes, there are. As an example, we can consider some of these.

.. There's a family of parallel lines here, which is kind of instructive to observe. And I'm talking about these truss beams of the bridge, this diagonal family of truss beams, and this other diagonal family of truss beams from the bridge. Let's extend them. And when we extend them, we see that this family of lines which is parallel in space does indeed converge to a vanishing point here, and this other family converges to a vanishing point here. So we have two very nice vanishing points off of the horizon line.

Now, we do have to imagine extending our picture plane a little bit in order to see them. So we have to imagine extending it up here so that this vanishing point is on our picture plane. And if we do that, then we can ask the question, why do the truss beams appear to converge to this vanishing point, v? Well, remember how we proved the vanishing point theorem? We said that if you take any line in space, follow it, follow it, follow it along with your sight lines, imagine the point line is infinite in space, you keep following it along, you'll eventually get to a limiting sight line which is parallel to that line in space but coming from your eye. So as we follow the truss beam along, and we have to pick one truss beam and follow it along, and we'll eventually get to a limiting sight line which is parallel to those truss beams in space but starting at your eye. And that's going to intersect the picture plane somewhere.

Because that family of lines was not parallel to the picture plane. So you will intersect your picture plane somewhere and that's this point, v. So now here's an important question we can raise. With our initial set of vanishing points that we associated to the ground plane, we asked what they were and we decided that they're images of points at infinity. And those points at infinity, we defined them precisely and associated them to families of parallel lines in the ground plane.

And that's how we built the extended Euclidean plane. But what are these new vanishing points? Are they also images of points at infinity? And if so, where do these new points at infinity live? They don't seem like they lie in the extended plane P2. If you imagine the ground plane here, extended so that it includes points at infinity, well, these don't look like they lie, that they're associated to it. And anyway, P2, the extended plane, already seems kind of maxed out on points at infinity. So the question is where do we house these new points at infinity? Where do we put them? We need more space.

What we need is the extended Euclidean space. So let's define this in much the same way that we defined the extended Euclidean plane. Namely, for every line L in R3, let bracket L denote the equivalence class of all lines in R3 parallel to L. This is the exact definition we did in R2, except now we're looking at lines in space. So bracket L is the collection of all lines in space that are parallel to a given line.

Another way of imagining that, take a line in space, you can translate it around, I wish I had a string, but you can take a line in space, I have a mouse, so hopefully you can see this line, you can take this line in space and you can translate it around in space all over the place. And those lines that you're seeing are all elements of this family in this equivalence class of L. You can't change the orientation, you can't twist it and turn it and do this kind of stuff, but you can translate it, preserving the parallelism class. So we denote the equivalence class of all lines in R3 parallel to L by bracket L, and let's define a pointed infinity PL for every class, every parallel class of L. So we just introduce a pointed infinity associated to every family of parallel lines, and we declare it to be incident to all the lines in that family, all the lines in R3 that are parallel to L.

So this is exactly how we built up the extended Euclidean plane, except now we're in three dimensions. So this takes care of all the new points at infinity. We get tons of new points, but this is all the points at infinity we could possibly want. But do we have to add any new lines in order to have some basic nice geometric features? Ideally, we'd like this extended Euclidean space, which we're building out of R3. I mean, essentially we want to add these points at infinity to R3 to get our extended Euclidean space.

But we want to make sure it's a linear space. It should at least have this property that any two points determine a unique line. Is that still going to be the case? Well, we've added lots of points at infinity, and we probably don't want to say that all of our points at infinity are collinear. There's just too many of them. So we have to be a bit more careful this time.

We need to group our points at infinity into various lines at infinity. In particular, for every plane pi in R3, let's let bracket pi denote the equivalence class of all planes in R3 that are parallel to pi. Again, you can imagine this as taking a plane in space, my phone here represents a plane, and translating it around in space. No rotating it, no turning it, just translating it around to get other planes that are parallel. In space, that collection of translates of a plane, that collection of all is the collection of all planes parallel to that initial plane.

And that is bracket pi. It's a set of all planes parallel to pi. It's the equivalence class of pi. So given that class bracket pi, let's define a line at infinity, L bracket pi. And we're going to define it to just be the set of all points at infinity. We already have points at infinity running around everywhere.

Every family of parallel lines has a point at infinity associated to it. We want to organize those. So the line at infinity L pi is the set of all points at infinity associated to lines in pi. We have our plane pi here.

It has lots of lines on it. Each of those lines has a point at infinity. We're grouping all those points at infinity up into one line at infinity called L pi. And we do this for every class pi. So it just consists of all the points at infinity associated to lines in pi. In fact, however, equivalently, we can also notice that L bracket pi, it also just consists of all the points at infinity associated to lines parallel to pi.

That's saying the same thing. Because here's pi, if we have a given line on pi and we have another line that's parallel to that line on pi but off of pi, I take that line and I translate it off to somewhere else. Well, this line on pi and its translation off of pi are going to have the same point. They're going to share a point at infinity. The point at infinity associated to this one is the same as the point at infinity associated to this translated one. So it means the same thing to say that L pi, this line at infinity L pi, consists of all points at infinity associated to all lines parallel to pi.

If this seems a little abstract, let's try and visualize it. So let's go back to some drawings of lines in space and to this painting. And let's look at an example. Let's let L be the physical truss beam in space, which is represented on the picture plane by this red line.

So L is a line in R3. M is another line in R3, which in the picture plane we're seeing as this blue line here. So there are images of these red and blue lines in the picture plane, this one and this one. So in this case, the point at infinity associated to L, its image is this vanishing point, is this point here. So PL, we see the image of PL right here. And indeed, all of these lines, red lines, although they're parallel in space, they share this point at infinity PL.

And in the picture plane, they appear to converge to this vanishing point, which is the image of PL. Similarly, with the blue lines, they're all parallel in space, so they share a vanishing point PM. And we see them in the picture plane as converging to this point here, which is the vanishing point, which is the image of PM. So we can see these two points at infinity. And these are not points at infinity on the horizon line.

They're off of that. So in particular, they're not on this line at infinity. But they do share and determine a line at infinity here, this green line L bracket pi, where pi I'm defining to be a vertical plane in R3, which contains the truss beams. So all of these red and blue truss beams, they're all coplanar in space. And if you want to imagine that plane, it's just a vertical plane. To help you imagine it, it's a vertical plane, and it's perpendicular to our picture plane.

So they all live in that plane pi. So an L pi, let's go back to the definition for a second. L pi consists of all vanishing points associated to lines in pi. So it consists of all vanishing points associated to lines in pi. These truss beam lines are in pi, and these truss beam lines are in pi. So L pi must contain this vanishing point at this point at infinity and this point at infinity.

And we see the image of L pi as this line in the picture plane. In fact, there's another line at infinity whose image we see in the picture plane. So if we let R2 denote the ground plane within R3, then the yellow horizon line is just the image on the picture plane of the line at infinity, LR2. So LR2 is a line at infinity consisting of all of the points at infinity associated to all lines parallel to the ground plane, just as we defined it earlier. And in this new definition as well, it coincides with our old definition.

We get this particular line at infinity, whereas earlier that was the only line at infinity that we defined when we defined the extended Euclidean plane. Now it's just one of many. And it really is one of many. So just as the ground plane and the vertical truss plane determine LR2 and L pi, any family of parallel lines in R3 will determine its own line at infinity. Now we'll actually be able to see most of these lines at infinity in the picture plane just as we saw L pi and LR2.

Most other families of parallel lines will have a line at infinity, which in the picture plane will look like a line. So there is an exception. If the family of parallel lines is parallel to the picture plane, then, so that's a small typo here, it will still determine its own line at infinity, but that won't be visible in the picture plane because those planes are parallel to the picture plane. So we won't get to see it from this perspective view. So there's lots and lots and lots of lines at infinity because there's lots and lots and lots of families of planes.

Now every plane pi in R3 has an associated line at infinity. So it's very natural to just consider the union of pi with that line at infinity L bracket pi as a single entity, just like we did when we created the extended Euclidean plane from R2. So when we do that, let's give it a special name. Given a plane pi in R3, we'll define its projective extension P of pi to be the linear space consisting of points pi union L bracket pi. And this construction will come in quite handy in the future.

Given any plane in R3, we can just extend it, add on all the points at infinity associated to it to get an extended plane living in this larger extended Euclidean space, which we'll now define. So let's define the extended Euclidean plane, sorry, extended Euclidean space. So we'll denote the extended Euclidean space by P3, and it's defined to be R3 union the collection of all points at infinity associated to all lines in R3. That's how it's defined as a set. But in addition to this set structure, we want it to be a linear space, so we need to make sure there's enough lines to give a unique line between any two points.

So I claim that after including lines at infinity, all of the lines at infinity, all the Lpis for all planes pi, this will turn out to be a linear space. But I'll leave it as an exercise for you to verify this. To prove that P3 is a linear space, you have to show that any two points determine a unique line. But it's best to divide that into a few sub-cases. First that any two ordinary points determine a unique line.

That shouldn't be too difficult. But then that any two points at infinity determine a unique line. And finally, that an ordinary point and a point at infinity will determine a unique line. If all of these three are satisfied, then P3 is going to be a linear space. So as another important remark, from now on, whenever we're dealing with a three-dimensional space, or whenever I just say the word space, or three-dimensional, let's assume I'm referring to the extended Euclidean space, not the ordinary Euclidean space R3. Unless I very specifically say we're talking about the ordinary Euclidean space R3.

In other words, whenever I say 3D space, we'll assume that we're not just talking about R3, but we also have lots and lots and lots of points at infinity running around. And when we refer to a plane pi in P3, or a line L in P3, we'll also by default mean they're extended versions. So when I say a line in P3, I'm not just talking about the ordinary line. I'm talking about the line plus that point at infinity that's associated to it. When I say a plane in pi in P3, I'm talking about a plane union all of the points at infinity associated to it, to all of the lines in it.

So unless we specifically state otherwise. And I want to, this is an important exercise, the intersection of two distinct planes in R3. Given two planes in R3, they might intersect in a line, but they might also be parallel and have an empty intersection. So a question I want to ask is, what are the possible intersections of two distinct planes in this extended Euclidean space? As I just remarked, when I say two distinct planes, I mean they're extended versions. So each plane includes many, many points at infinity, a full, oops, that's a typo, many, many points at infinity.

The whole line of infinity is associated to each plane. So I have one plane with its line at infinity. I have another plane with its line at infinity. Where do they intersect? What does their intersection look like? That's the question. And I'll give you the answer as a hint.

Their intersection is guaranteed to be a single line. So can you show this? Can you show that any two planes in P3 will meet in a line? Basically, you have to worry about the case where the ordinary Euclidean versions of the planes, their restrictions to R3 are parallel. In that case, do they still meet in a line? As another hint, it has something to do with the fact that the line at infinity for this one is actually the same as the line at infinity for this one. Because these planes are parallel, any point at infinity associated to this, you would just take that line that's connected to that point at infinity, a translator that will be in this plane, and will have the same point at infinity. So if they're parallel in R3, the associated lines at infinity will be the same, and their intersection will be that line at infinity. So you can take a moment to think about it a little more and convince yourself, but that's the basic idea.

So I want to just note that we witness points at infinity both on and off of the horizon line all the time. Like when we see a representation of a staircase, we're seeing lots of points at infinity on the horizon line. It's all of these points on this horizon line here. But we also see points at infinity off of the horizon line. Like these diagonal lines that are following the corners of these stairs will eventually intersect somewhere up here.

So we have one there. Similarly, these stairs here will give us another point at infinity somewhere here. So we're actually very used to seeing points at infinity that lie not just on the horizon line, but in all kinds of other places. When we see a photograph of a building in three-point perspective, or any object in three-point perspective, that third point is off of the horizon line, because the vertical lines that converge to it are not parallel to the ground plane.

And as a fun exercise, this painting, this drawing by M.C. Escher, includes both staircases and three-point perspective. So you can see how many vanishing points you can find. I don't think I could even find all of them, but there are many, many vanishing points associated to this picture. So you can see how many of them you can identify. And this staircase you might notice is in fact, you can continue going up it forever or down it, down the staircase forever.

There's also a paradox rolled into here. And maybe in the process of exploring these vanishing points, you can get a better sense of how this paradox is achieved. So I'll leave that as a challenge. The last thing I want to say is that perspective drawing, it reveals many more points at infinity than we thought before. Not just on the ground plane, but literally everywhere in all directions. Every family of parallel lines in space has a point at infinity associated to it, which we actually get to see by viewing it in perspective.

So in this way, perspective drawing allows us to view portions of the extended Euclidean space P3.