Our Mathematical Senses

The Geometry Vision

Prof. Vijay Ravikumar

Department of Mathematics

Indian Institute of Technology- Madras

Lecture-32

Chapter Three: The Shadow of a Square

Russell and Jeffy incredible incredible I do not believe it! Hmmm, lovely. I think this works out. I do not believe it! Terrible, terrible! But try it! Picture of the picture! Welcome to the Geometry of Vision, week 3. We're now experts at drawing squares from various perspectives. And you may have noticed that drawing the image of a square is actually quite similar to casting the shadow of a square. Or to the pinhole projection that happens when you take the photograph of a square.

The set of ways a square can appear in the picture plane is the same as the set of ways its shadow can appear under projection from a single point light source. Like the LED torch light on your phone. So what is that set of ways that a square can appear? In other words, what can the shadow of a square look like? And what can't it look like? Can the shadow of a square look like a circular disc? Can it look like a map of India? Can it look like a rhombus or a long rectangle? Or what about this arbitrary quadrilateral? Here's another way of asking the same question. What kind of transformation of the plane does a perspective shift induce? In other words, if we think of a perspective shift as a function from the plane to the plane, then what exactly is that function? Is it similar to any other transformations you may have encountered? Answering this question will help us precisely formulate what changes and what stays the same when we shift perspective.

In order to do that, let's briefly review the concept of a transformation of the plane by looking at examples you may have seen before. Perhaps the simplest transformations of the plane are those that preserve length, also known as isometries of the plane. These include vertical translation maps, like this one, or horizontal translation maps. Or we can combine these to get an arbitrary translation by a vector AB. The rotation maps are another set of transformations that preserve length.

For any angle theta, we have a map that rotates the plane counterclockwise about the origin by that angle theta. Finally, there's one last distance-preserving map, reflection. Reflection about the y-axis, for example, which sends every vector x,y to the vector negative x,y. Because they all preserve distances, all of these transformations take the unit square to another square of the exact same size. Now, what if we relax our requirement that our transformation of the plane preserves length? What transformations of the plane have you seen which distort length? One of the simplest is the dilation, or stretching map, which takes a vector x,y to a vector lambda x,mu y.

And these dilation maps take the unit square to various rectangles, or to larger and smaller squares if the scaling parameters lambda and mu are equal. Are there any other maps of the plane that you've seen before? If you've taken a course in linear algebra, you might have seen a shear transformation, which takes a square to an arbitrary parallelogram of your choice. Here's one example of a shear transformation, though we'll see some other types later on. Notice, however, that all of these maps, translation, rotation, dilation, reflection, and shear, they all take the square to some parallelogram. On the other hand, this quadrilateral, from our third perspective view of the tiled floor, is definitely not a parallelogram.

So the perspective-shifting map that created it must be something entirely different. Ah, excuse me, excuse me! For those of you who are interested in the interaction, a concept that you might find the most illuminating is that of central projection. That is exactly the perspective-shifting map you are talking about. The ray of light travels from the domain plane to the image plane, and a point from which the light emanates. That is called the center of perspective.

That's correct, and when you draw on a picture plane, it's just like central projection, but in reverse. The light travels from the domain plane to the image plane to your eye, and your eye is the center of perspective. Hold on a minute, this is very interesting! I just realized it is a third possibility! When you take a photograph, the center of perspective lies inside the shuttle of the camera, and is in between the domain plane and the target plane of the path of the light. That's correct. In this case, the target plane is actually the photographic negative on which the photo is recorded.

If you've ever seen a photographic negative, the orientation of the image is reversed, precisely because the center of perspective lies in between the domain plane and the image plane. Mathematicians define a special map called a perspective, which captures all three of these real-life projections. In order to define a perspective, fix two planes, P and Q, as well as a point O that lies off of both planes. A perspective centered at O from plane P to plane Q will then linearly project points from P to Q with respect to the center of perspective O. In other words, it'll map a point X on P to a point Y on Q, whenever O, X, and Y are collinear.

Now the resulting perspective from P to Q will behave like a shadow projection on one region of P. A perspective drawing on another region. And a photographic negative on the remaining region. However, you might have noticed that this perspective is not actually defined at every point of the plane P. For example, the point X will not project to any point on the plane Q, since the line X, O is parallel to Q.

We'll soon find a way to fix this issue, but in the meantime, here's a puzzle. What are the possible images of a single square under a perspective? Can you produce any quadrilateral, or are there some that you can't produce? And can you get shapes that are not quadrilaterals? See if you can figure this out using one of the real-life examples of perspectives. For example, you could use the LED torch on your phone to cast shadows of a square piece of paper, and make a conjecture about which shapes are possible to produce. Or you could also use your phone's camera to take photos of that square, and see which shapes you can produce as images of that square. To get started, try producing a trapezoid, a rhombus, or a long rectangle.

Are you able to produce all these shapes, or are some of them more difficult than you might have expected? If you're working with shadows, make sure you pay attention to what kind of light you're using. Is it coming from a candle or a bulb, or is it coming from the sun or the moon? Believe it or not, it makes a difference. This is because the sun is so far away that the light rays approximate parallel projection, not central projection. And they may create slightly different shapes. Once you've come up with a conjecture regarding the possible shadows of a square, write it down.

We'll go through the solution next week, and you can see if your conjecture holds up. In the process of solving this problem, we'll refine our understanding of perspectives, which will help us prove Pappus's theorem from last week, and also finally answer the question of what changes and what stays the same under a perspective shift. So let's get started.