Our Mathematical Senses

The Geometry Vision

Prof. Vijay Ravikumar

Department of Mathematics

Indian Institute of Technology- Madras

Lecture-33

Video 7A: perspectivities the maps that shift perspective

So, let's take a closer look at perspectives, which are the maps that shift perspective. So, this week we're focusing on the question, what are the possible perspective images of a square? What are the shadows of a square? Or what are the perspective views of a square? They're really the same thing. And one of the reasons we're interested in that question is that it's intimately related to a bigger question, namely, as a transformation of the Euclidean plane R2, what does a perspective shift look like? So, let's take a closer look at the perspectivity map from the introduction video for this week. So, the definition is the following. Let's suppose that pi and pi prime are distinct planes in R3. Here's pi and here's pi prime.

And o is a point that's not contained in either plane. Now, a map gamma from pi to pi prime is a perspectivity centered at o. It's called a perspectivity centered at o if for any x in pi, the points o, x, and gamma of x are collinear. So, in other words, if gamma relates x to the point in pi prime, the point in the plane pi prime, which is pierced by the line o, x.

So, to see another example, y is another point in pi and the line o, y hits pi prime at gamma of y. So, gamma is relating y to this point here, pierced by o, y. And similarly, z is here and gamma of z is here. And o, z, and gamma of z are collinear. So, let's see another example of a perspectivity, which is where the planes and the point are arranged in a slightly different way in space.

And over here, pi is this plane, pi prime is this plane, and o is centered here between the two planes. And in this case, we see that x, o, and gamma of x are collinear, y, o, and gamma of y are collinear, and this point z, o, and gamma of z are collinear. So, in, oops, gamma of z is here, o and gamma of z. So, again, we have the same relationship. Gamma is somehow linearly projecting points from pi to pi prime through the center of perspectivity o.

But there's a natural question that we should raise here, which is whether we actually get a well-defined map from the plane pi to the plane pi prime. We certainly do in some parts of pi. That's what we're seeing in these images here. So, this portion of pi seems to very naturally project to this portion of pi prime, and it looks like this map is well-defined. But what happens when the line through o and x is parallel to pi prime? In other words, the line through o and x doesn't hit pi prime at all.

So, to see an example of that, let's extend our plane pi, make it bigger, just imagine extending out in space, and let's pick a point x on pi such that o x, the line o x, is actually parallel to this plane pi prime. So, o x now is not going to hit pi prime anywhere. The line through o and x will not meet pi prime. And, as a result, this map, this map gamma, the perspectivity, doesn't seem to be defined on x. It's not going to map x anywhere on pi prime.

And that was this x, but we can also just look at another x over here, which is all such that o x is again parallel to pi prime, or at point x over here, again o x is parallel to pi prime. There's multiple choices of x such that o x is parallel to pi prime. In fact, there's an entire line, this line here, on pi, such that, on which the map gamma is not defined. What is this line? Well, it consists of all the points x in pi, such that the line connecting o to x is parallel to pi prime. Another way of arriving at it, or imagining it, it's just the line we get if we were to translate pi prime through space so that it passes through o.

So that translation of pi prime up to o is going to hit pi at a line, this line here. So that's how you can get that line of points such that o x is parallel to pi prime. So now, the map is not really defined between planes in Euclidean space. Pi and pi prime are two planes in Euclidean space, and unfortunately the perspectives map is not really well defined, as we've given it. But what about if we extend our space to p3? We include points at infinity.

So let's again take this point x1 here and consider the line o x1. Now, in p3, the line o x1, it's going to meet a point at infinity. We keep following o x1 out, and eventually it'll meet a point at infinity associated to it, p o x1. So o x1 is parallel to pi prime in R3. So the extension of pi prime, remember, for every plane in R3, when we defined the extended space p3, we gave it an associated line at infinity.

And the extension of pi prime will also contain the point at infinity, p o x1. Why is that? Well, o x1 is parallel to pi prime, and by definition, pi prime is going to... The line at infinity.

.. The line at infinity, l pi, that's associated to pi... Sorry, l pi prime, that's associated to pi prime in p3, that's going to consist of all the points at infinity of all the lines parallel to pi prime. So in particular, it's going to include p o x1.

Similarly, o x2 will meet a point at infinity, p o x2. And that will also be contained in the extension of the plane pi prime, simply because it will be contained in the line at infinity associated to pi prime, simply because the line o x2 is parallel to pi prime. It's parallel to this plane pi prime. And finally, we get the same thing with o x3. Again, o x3 is parallel to pi prime, so its point at infinity will also be contained in the extension of the plane pi prime in p3.

We're working in p3, which is why all of this now holds. So all of these lines really do intersect pi prime, as long as we're working in p3. So let's go back to the definition.

We said we... I wrote that pi and pi prime are distinct planes in R3, and we saw that the definition doesn't quite work out for us. But if we change that to p3, now everything seems to work out much better. The map is well defined. For any x that we've chosen so far in pi, o x will meet pi prime somewhere, either at an ordinary point or at a point at infinity of pi prime. And I just want to apologize and remark on this slightly confusing notation, because right now I'm using pi and pi prime to refer directly to the extended plane to the extended planes in p3.

Whereas earlier, when we were talking about R3, I was using the same notation, pi and pi prime, to refer to just the ordinary planes. And we're going to keep doing this. We're going to keep using these Greek letters to refer to planes. And depending on the context, the plane could be in R3 or it could be in p3. So I'll try and be as explicit as I can about which context we're in, both in terms of what I write and what I say.

But just keep that in mind. On its own, pi could be referred to just the ordinary plane or the extension of that ordinary plane in p3. And this seems great, but I just want to make sure we're not getting lost in this extended world with points at infinity and lines at infinity. And ultimately, we want to know about transformations of the ordinary Euclidean plane R3. So we are going to get back to that, but we do need to work with these extended planes in p3 in order to understand transformations of R2.

So I just want to reassure everyone that we will get back to R2 and understand the transformations of R2, finally, at the end of this journey. So eventually, not only will we return to R2, but we'll also arrive at some quantitative answers to the question we asked of what changes and what stays the same under a perspective.