Our Mathematical Senses

The Geometry Vision

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Lecture-34

Video 7B: Visualizing perspectivities

How do we define perspectives? Now that we've defined perspectives, I want to spend some time trying to visualize them better. In particular, we've defined perspectives between planes in the extended space P3. But it's still very interesting and valuable to visualize their restrictions to planes in R3. So let's look at some real-life examples or interpretations of perspectives. As a warning, we're going to keep abusing notation and letting the symbol pi refer to both the ordinary plane in R3 and an extended plane in P3. It will hopefully be clear from the context.

So right now, let's imagine that we're in R3 for the time being. And pi and pi prime, they really are planes in P3, but we're looking at their restrictions to R3. We're ignoring all the points at infinity for the time being that are associated to pi and that are associated to pi prime. Actually, they also have associated points at infinity, and each one has to have a line at infinity associated to it.

But we're going to ignore that for the time being, and we'll bring those in in just a second. So let's imagine a set of railway tracks on the ground plane pi. And imagine that we are drawing on a picture plane pi prime, which is perpendicular to the ground and to the railway tracks. And our eye is kind of a center of perspective, O, as well. So here's a point x on pi, and it maps to a point gamma of x on pi prime.

And our sight lines create a perspective image of a portion of pi upon a portion of pi prime. So the portion of pi prime over here, these railway tracks are creating an image of railway tracks in this portion of the plane pi prime. And we can ask the question whether we can extend the drawing further once we've drawn the tracks up to this vanishing point here. And unfortunately, it doesn't seem like we can if we're imagining that we're actually in real life doing a drawing of the ground plane. Of course, there might be stuff sitting above the ground plane, but for this example, everything we're drawing is in the ground plane pi, and we're drawing onto this plane pi prime.

So no matter how far out we look in the plane pi, no matter how far out we look, we're not going to see anything above or even on the horizon line H, no matter how far out we look. So this region, which is kind of infinite, is going to map to this finite region here on pi prime via the perspective. And similarly, when we look down, we can't draw down beyond the ground edge. We can draw all the way down to this ground edge where the picture meets the actual railway tracks, but we can't draw beyond that. So that's kind of an upper limit and a bottom limit of our drawing interpretation.

So basically, the perspective map, if we interpret it as a drawing, it's stuck there. But in fact, the map is defined everywhere, so it does extend further down. But we'll need to change our physical interpretation in order to extend it, in order to understand that part of the map. So let's try drawing further down, and now, I mean, let's try exploring what the perspective does further down. But we're not drawing anymore.

We're not doing a perspective drawing. We're now doing kind of a shadow projection of a different portion of pi to a different portion of pi prime. So now this point here is going to get projected, this is a point from pi, but it's going to get projected to this point here in pi prime. And similarly, we can keep projecting all these points, and we'll extend the image of the railway tracks further down that way, through that projection. We can keep going further down, we can keep projecting, doing this shadow projection, but once again, we're going to reach a limit of that interpretation.

And when we reach this line k in the plane pi, then suddenly the lines joining O, just to clarify, this is the center of perspective O right here, and draw it in black. So once we reach that line k, suddenly the points connecting O to points in k are all going to be parallel to the plane pi prime. They're going to be parallel to that vertical plane pi prime. They're not going to hit pi prime as a result. So we're reaching a hard limit because suddenly in the real life R3 interpretation, our perspective doesn't even seem to be defined.

We're actually hitting points at infinity of pi prime now. This is another region, this is a finite region of pi, which is going to map to this infinite region of pi prime. So we see that a perspective kind of mixes up finite and infinite regions. It doesn't really care about what's finite in one plane or infinite in another. Somehow it's thinking in a different way.

Now beyond the limit, beyond the line k, we need to change our real life interpretation again. And suddenly we're not doing shadow projection, we're taking a photograph and recording the image on a photographic negative. So we're taking a photograph of the ground plane pi and recording the image on this photographic negative pi prime. So this point here is going to map to this point here, this point here is going to map to some point over there, and so on. We'll get this image here.

And you might notice that in this region, the perspective seems to be reversing orientation, at least from our R3 vantage point. So this brown line over here is suddenly on the other side of the railway tracks here. It's always on the right side in this image, but suddenly here it's on the left side. So we've reversed orientation when we've gone from this region to this green region here. So this is the third region, and in this case both actually seem infinite in both pi and pi prime.

So maybe the moral of the story is just that perspectives are kind of strange if we try and give them a global, real life interpretation. By global I mean looking at its behavior on all of pi and all of pi prime, not just a small portion. It sometimes looks like a camera, sometimes looks like a flashlight, sometimes looks like our eye doing a perspective drawing. So on the other hand, but to a projective geometer working in P3, whose primary setting is P3, all these interpretations are actually just a red herring. They're misleading, because the map is defined without reference to time or light traveling or an order of points being arranged.

More specifically, let's just recall the definition. So x, a point x, maps to gamma of x, x in pi maps to gamma of x in pi prime, if and only if the points O, x, and gamma of x are collinear. There's nothing there about a light ray from O first hitting x and then hitting gamma of x, or first hitting gamma of x and then hitting x. That doesn't really come into play. So in fact since we're in P3, this line which connects O, x, and gamma of x is actually circular.

It's going to have a point at infinity, P, O, x, and via that point at infinity it's actually kind of circular. So there's really no preferred or natural ordering to the points O, x, and gamma of x. We can take them in any order, depending on where we start and which direction we go. So to a projective geometer all these interpretations are really a red herring. They're not telling us anything.

They're kind of misleading us in a different direction from the actual fundamental nature of this map. But at the same time, at the end of the day, we do inhabit R3 and understanding the restriction of a perspective to R3 is still a very valuable exercise. So it's sometimes good to put on both hats and see it from both perspectives.