Our Mathematical Senses

The Geometry Vision

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Lecture-35

Video 7C : Perspectivities are well defined

Now, although we've defined perspectives and seen some examples, we still haven't actually shown carefully that project that perspectives are well defined. So let's do that now. And first, let's see what I mean by this. What have we missed? What have we not carefully shown? So let's let gamma o. o is a center here. Let's let gamma o from pi to pi prime denote the perspective centered at the point o from a plane pi in P3 to a plane pi prime also in P3.

And we haven't shown that this is a well-defined map. And to do that, we need to check that for any point x in pi, the line o x intersects pi prime at a unique point. So it looks like it's true. It looks like if you have a point for any point.

Let me choose a better point for any point in pi. We seem to get a unique point in pi prime, where the line o x intersects pi prime. But have we really shown that this is true for any point? Remember, we're working in P3. So we have to be a little careful about all the points in infinity. And it might be helpful to prove a slightly more general feature of P3.

So let's let pi be any plane in P3. And let's let l be any line in P3, which is not contained within pi. Then I claim that l and pi are incident at exactly one common point. In other words, the line l will pierce the plane pi in one point, and only exactly one point, unlike the situation in R3, where a line l and a plane pi can be parallel and not meet at all. The claim here is that disjointness cannot happen in P3.

So to prove this, let's first assume that l is not a line at infinity. Let's say that l is just an ordinary line. It'll still have a point at infinity, but it won't be a full line at infinity, consisting entirely of points at infinity. In other words, we can consider the restriction of l to R3 in this case. And we can also consider the restriction of pi to R3.

So let's look at their restrictions to R3. And in those restrictions, they're either going to intersect at an ordinary point in which we're done, or they'll be parallel in R3. But if they're parallel in R3, what's going to happen? Well, in that case, l will contain a point at infinity, pl. And since l is parallel to pi, that means that there's a line here that's parallel to l sitting in pi. And it's also going to hit the same point at infinity, pl, because pl is the point at infinity connecting all lines in this family of parallel lines.

So in this case, the plane pi and the line l are going to meet at the point at infinity, pl. They're going to meet at exactly one common point again. So we're kind of done. In all these situations where l is not a line at infinity, it's clear that this statement holds. On the other hand, suppose that l is a line at infinity, l tau, for some plane tau.

So there's some other plane tau. And l is the line at infinity associated to it. It consists of all the points at infinity of all of the different families of lines parallel to tau. Now, we know that l tau is not contained in pi. We've assumed that, because l and pi are sorry, that's what our assumption is, that l is a line in P3 not contained in pi.

So l tau is not contained in pi. Therefore, we know that tau is not parallel to pi. Because if tau were parallel to pi, they would share the same line at infinity. So therefore, tau is not parallel to pi, which means that tau and pi are going to meet at a line h. And we can see it a little more clearly if we extend pi out.

We'll just look at this portion of pi instead. And we see that pi and tau meet at this line h. And what can we say? Well, h, this line h, is going to have a point at infinity, ph, associated to it. And since h is in tau, that ph must lie in the line at infinity l tau. But ph is also going to be a point at infinity of this plane pi, because h is in pi.

So that means that l tau, the line at infinity l tau, will meet the plane pi at this point at infinity ph. So once again, our statement holds. l and pi are incidents at exactly one common point, ph. So it seems like we've kind of proved the statement. We're basically done.

We've kind of considered all cases. l can be a line at infinity. It could be an ordinary line. What else is there? But there's a confession I have to make, which is that there is a final case to consider. And this is because I've been a bit lax in defining the extended plane P3.

So if you remember, in Euclidean geometry, there's a very standard property that any three non-collinear points in R3 will determine a unique plane in R3. And we're really going to want that property to hold in P3. That's a very fundamental property. But does it hold? Is it true? And unfortunately, the way we've set things up so far, it doesn't. Because we could take three non-collinear points at infinity, and then we're going to have a problem.

So we need to add an additional plane, much like how when we built up the extended Euclidean plane P2, we added all these different points at infinity. But then we wanted this property to hold that any two points at infinity should determine a line. And that wasn't working. So we had to add a line at infinity in order for that basic property to hold. So analogously, in P3, we've added all these points at infinity.

We've added all these lines at infinity. We've organized those into lines at infinity. But now we need an extra plane at infinity to contain all those lines and points at infinity. So let's define the plane at infinity, pi infinity, to just be the collection of all lines at infinity. It just consists of, or another way of saying it, it contains all the points at infinity.

So actually, what I've written here is actually not technically correct. Oh, no, that is correct. That is correct. Sorry. It consists of all the points at infinity PL.

It's not the set of lines at infinity. It's a set of all points at infinity. But of course, it contains all the lines at infinity as subsets of it. So this is the plane at infinity, pi infinity, just the collection of all points at infinity. And if L is a line not contained in pi infinity, then, so let's finish this statement.

Let's assume that our plane, pi, is pi infinity, the plane at infinity. Let's let L be any line in P3 not contained in pi. Well, in that case, it's going to meet pi prime at exactly one point, its own point at infinity. And if L is not contained in pi infinity, it's not a line at infinity. It's not going to be, it has to be an ordinary line.

So it'll have one point at infinity, PL, and it's therefore going to meet pi infinity at exactly that one point. So that does it. That proves this statement, which is a fundamental incidence property in P3. So coming back to our example, can we see where gamma O from pi to pi prime is going to send the line L at infinity, L pi? So where is it going to send the line at infinity associated to the plane pi? Well, let's just look at where it sends some of the points in L pi.

Here's a point, PL1. It's associated to this line, L1. L1 is parallel to pi. So this is indeed a point from L pi. And where does O send it? Well, let's connect O to there and see where that line pierces the plane pi prime.

So it gets sent to this point here. Similarly, here's another point at infinity from a line parallel to pi. This is another point at infinity, a point within the line L pi. And it's going to get sent to this point here. This point at infinity is going to get sent to this point here. So all of these points at infinity are getting sent to various points along this line, H.

And in fact, the entire line at infinity, L pi, will get sent to the line H in pi prime. So the line at infinity is actually getting sent to an ordinary line, H, in pi prime. So that's kind of interesting. The perspectivity can actually interchange, or it can turn a line at infinity into an ordinary line. And in fact, it can even do the reverse.

So let's look at this line, K. That's an ordinary line in pi. But it's going to get sent to a line at infinity in pi prime, the line at infinity, L pi prime. So you can see that, for example, this ordinary point gets sent to this point at infinity, PL1. This ordinary point gets sent to this point at infinity. And this ordinary point gets sent to this line at infinity, PL3.

All of those are points at infinity in pi prime, specifically lying on the line at infinity, L pi prime. So K, an ordinary line, got sent to a line at infinity. So gamma O will interchange ordinary points and points at infinity. It doesn't actually distinguish between them.

So H is really gamma of L pi. And K, sitting in pi, is the pre-image gamma inverse of L pi prime. So actually, we can say a little more. Gamma O turns out to be a bijective map between the planes pi and pi prime when it's considered in P3. Bijective means it's one to one. So each line L through O in P3 relates a point of pi with a point of pi prime.

So every line through O, we can draw any line through O. And it's going to relate any line through O is going to relate a point of pi with a point of pi prime. So most lines L through O are going to relate ordinary points of pi and ordinary points of pi prime. Those are a little easier to visualize. But some of the lines through O, namely the ones that are parallel to pi prime or parallel to pi, are going to relate ordinary points to points at infinity or points at infinity to ordinary points or maybe points at infinity to points at infinity.

So if L is parallel to pi, for example, like this L here, well, that's going to relate a point at infinity of pi to an ordinary point of pi prime. On the other hand, if L is parallel to pi prime, then that's going to relate an ordinary point of pi, namely L intersect pi, to a point at infinity in L prime. Finally, it can also happen that L is parallel to both pi and pi prime. In other words, it's going to be parallel to their intersection.

It's going to be parallel to this line here. And you can kind of imagine it like that. And in that case, it's actually going to fix. So if this is L, then the point at infinity, P L, is going to also be in the intersection of pi and pi prime. And therefore, it's actually going to be fixed by this perspective, gamma O. Gamma O anyway is fixing all the points in the intersection of pi and pi prime.

So in particular, it'll also fix this point at infinity of this line of intersection. So in fact, we can say something even stronger. So this kind of shows that it's a bijection. It's a bijective map between these two extended planes sitting in P3. But remember, pi and pi prime are not just subsets in P3.

They're linear spaces. They're not just sets of points. Those points, certain subsets of those points, we're also building into lines. Certain subsets of those points are actually called lines. So gamma O is actually a map of linear spaces.

In particular, it takes lines to lines. It maps a line in pi to a line in pi prime. That's what I'm claiming. We can see many examples just in this image right over here. You can see that this brown line maps to this brown line. Each of these lines that are the railway ties, you can check that those map to railway ties up here.

The side rails, like this side rail, maps to this side rail. This side rail maps to this side rail. So you can see many examples how it takes lines to lines just in this image. But we should still be a little more careful in proving that it truly maps any line in pi to a line in pi prime.

So to show that, I'm leaving that as an exercise. But as a hint, it helps to examine the various cases individually. So let's just look at this brown line here, which maps to this brown line in pi prime. How do we see that? Well, one way to see that is that this brown line here, if we look at all the lines connecting O to the various points in this brown line, that collection of lines through O is going to span a plane. And that plane is going to hit, it's going to intersect pi prime in a line, namely this brown line. So in some sense, that's why any line here gets taken to a line here.

That's the version of that when these are ordinary lines. So we just have to generalize that claim. So in particular, as I just said, a point and a line determine a plane in R3 and in P3, provided they are disjoint. So that's the first thing you have to show, that given a point and a line, a disjoint point and line in P3, they're going to determine a plane, just as they did in R3. And secondly, we'll have to show that any two planes in P3 will in fact intersect in a single line. So as an exercise, can you show both of these statements in P3? Meaning you're going to have to consider points at infinity and lines at infinity, and even the plane at infinity.

So you'll have to consider those as individual cases. But each of those cases, once you restrict your attention to them, should follow naturally. But it's still a very useful exercise to try on your own. So good luck.