Our Mathematical Senses

The Geometry Vision

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Lecture-36

Video 7D: the harmonic conjugate theorem revisited

We now have all the tools we need to patch up a small gap that we left in the proof of the harmonic conjugate theorem back in the previous chapter. So let's try and take care of this unfinished business. So basically in the previous chapter, we cheated a bit when we claimed that the harmonic conjugate is well defined. This was the first coincidence we introduced, if you remember, the harmonic conjugate theorem. So just to refresh your memory, how did we prove..

. So the harmonic conjugate is this point D. In other words, the notation was that HAC of B is equal to D. HAC of B is the harmonic conjugate of B with respect to A and C.

And the way we defined it was given A, B, C, and D, we chose a point P in the plane, we chose a point Q on the line PB, we connected P and Q to A and to C. That gave us our points R and S, and that diagonal RS hit our original line at this harmonic conjugate point. Basically our construction involved creating this parallelogram PRQS, and viewing rather this quadrilateral, and viewing this quadrilateral PRQS as a perspective view of a parallelogram. That's how we proved that the harmonic conjugate was well defined. But we only looked at nice choices of P and Q in our proof.

We only looked at choices where it was easy to imagine this as a perspective view of a parallelogram. So what about not so nice choices? So let's take a not so nice choice and see if our proof can somehow still work. So let's choose any point P not on AC, like this one here. But now let's choose a point and let's connect P to B. Earlier we only chose points Q on the same side of this line as P.

But let's choose a point Q here. And remember, for a projective geometry, from the standpoint of projective geometry, even the idea of sides doesn't really make sense because this line is actually circular. So it's not that this line isn't dividing our line into two regions. They're still connected. So there aren't actually any two sides.

So it's only in our limited perspective view that it seems like there's two sides to this line. So let's choose a Q here on this other side. And let's continue our construction. Let's connect up A to P and Q and connect up C to P and Q. So, sorry, I'll do C first.

And then let's connect A to P and Q. Now, this gives us new points R and S. But they're not where we'd like them to be. R is here. R is the intersection of AP and CQ.

S is the intersection of AQ and CP. So it's over here. So when we connect them up, we get what used to be our diagonal of the quadrilateral. Now, it doesn't look like a diagonal. We get this line here.

And that's going to hit our original line at this point, HAC of B. OK, it's a little hard to see. But I kept the same A, B, and C as before. And you can maybe believe that it's actually the same point D that we had before. So it seems to still be well-defined.

The theorem still seems to hold. But our proof, our justification involving this perspective view doesn't seem so justified anymore. Earlier, we were interpreting PRQS. That's P-R-Q-S back to P. We were interpreting this as a perspective view of a parallelogram.

Now it's like a weird crossing X hourglass shape, which doesn't seem like a perspective view of a parallelogram that we could ever achieve in real life. But this is where we're going to bring in the machinery of the perspectivity. Because although it's not a perspective view of a parallelogram that we'll achieve in real life, it is the image of a parallelogram under a perspectivity. So we can still interpret it this way if we just enlarge our understanding a little bit. So let's extend these lines out a little bit.

Let's work in P3. And just for fun, we can shade in this region here. These two, these lines are, well, yeah, it's not, there's many regions we could consider. But let's just for fun use this one here. And that region is indeed the perspective, the image of this parallelogram in pi under a perspectivity centered at O. So you can imagine under this perspectivity centered at O.

Oops, sorry. My lines are really not very useful here, but you can get some points which, like here, it's a little flipped from this image, I guess. But basically, that's the situation. We have something here and something up here. So we'll get, you can play with this a little bit more. And you can convince yourself that the image of PrqS under a perspectivity is actually going to consist of something in this upper region here, because this is the green region, which maps to this green region.

And this part of the parallelogram lies in this yellow region, which is going to be projected down here. So in our real life interpretation, when we look at the plane pi, we will get an image that looks something like this. So in fact, our justification still works if you allow for perspectives instead of just narrow perspective views. You can also think of this parallelogram as kind of intersecting the picture plane. The picture plane, no, sorry, that's not true.

What I meant to say is that if this is the center of perspectivity, the parallelogram actually lies under you. If you're viewing it from here, it's under you. So it's not something we could do in real life.