Our Mathematical Senses

The Geometry Vision

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Lecture-40

Video 8D: projectivities as functions

We've been looking at projectivities between lines in space, but now I want to turn our attention to projectivities from a given line to itself. And I also want to look at a specific line, namely the x-axis in R2. Of course, R2 sits inside P2, but for right now let's just think about R2 and the x-axis within R2. And let's look at projectivities from that x-axis back to itself. And the reason I want to do that is because it'll let us think of projectivities as functions of a single variable x. So what we're going to see is that many familiar operations on the real numbers can be realized as projectivities from that real line, that x-axis, to itself.

So let's let the line L be the x-axis in the Cartesian coordinate plane R2, considered as a subset of P2. And let's look at projectivities from L to itself. So I've just marked a few points just to make it clear. This is the familiar real line x-axis within R2.

Here's the point 1, 0. Here's the point 0, 1. Here's the origin 0, 0. So let's start with an example. Let's look at this, at the perspectivity f01 composed with f02, where f01 is a map from L to this line m, y equals 1 half.

That's the line m. And f02 is a perspectivity from m back to L. And o1 is the center here, 0, negative 1. And o2 is the point here, 0, 2. So what is f01 composed with f02 going to look like? Well, let's just try it out on this point 1, 0.

Where is the point 1, 0 sent under f01? Well, let's just see. o1 will project the point 1, 0 onto the line m over here. So it's going to go to the point 1 and 1 half comma 1 half. Now where is this point going to be sent? By f02. Well, f02 is going to push this point down to this point here on the x-axis.

What is this point? Well, it's the point 2, 0. So we're sending the point 1, 0 to the point

2, 0. So 1 goes to 2. And yeah, that seems to agree with this function. x goes to 2x.

Of course, this doesn't fully verify that this is a construction of this function. But it at least agrees with this function on this one point 1. So as an exercise, you could try it out on a couple other points like 0, negative 1, or 2. And after doing a few more, you might convince yourself that this is indeed capturing this function on all points x. So I'll leave that as an exercise for you to verify this on a few more points.

But for the time being, let's look at another example. Oh, I'm sorry. Before that, let's look at a couple more points using this example. So remember that although this is R2, it's sitting inside of P2. And this line l has a point at infinity associated to it, which I'm just going to call denote as infinity.

So where is infinity getting sent by this projectivity? Well, l will hit this point infinity. But m is parallel to l. So it'll also touch the same point at infinity. And O1, when we look at the line connecting it to infinity, that's also parallel to these guys. So it's also hitting them at infinity.

So all three of these will hit at infinity. And infinity is actually fixed by this first perspective, FO1. On the other hand, FO2 is also going to hit both of these lines at that point infinity. It's also parallel. And so infinity is going to be fixed by FO2 as well.

So the composition is going to fix that point infinity. So it's a fixed point of this construction. And indeed, if you think about the function, this function will indeed fix infinity. If we're looking at the function x goes to 2x, that will in fact fix infinity. Doubling infinity doesn't change it.

So that brings up another interesting question. How many fixed points does this projectivity have in total? Are there any other fixed points? So it might be useful here to look at the function. What are the fixed points of x goes to 2x? Well, non-zero numbers all change. They all get doubled. But zero is sent to zero.

So we do have another fixed point. And it's easy to convince yourself that the construction actually fixes zero. FO1 will send zero, zero to zero, one half. Then FO2 is going to send zero, one half back down to zero, zero. So zero, zero is fixed.

So there's two fixed points. Now in general, if a projectivity from a line to itself has two fixed points, we call it a hyperbolic projectivity. That's just a technical term for it. But it means that it has exactly two fixed points. Let's look at a second example.

This is a totally new example. Again we have our line L. Again, I'm taking the same line M equals one half. Let me just write that so it's clearer.

Y equals one half. That's our point M. But I have two new centers, O1 and O2. O1 is equal to one half comma one. And O2, which is right here, is equal to negative one half comma one. So I claim that this composition gives a projectivity which constructs or which recreates the function X goes to X plus one.

So let's verify that. Let's try it out on a couple examples. So let's look at this point zero, the origin. It ought to get sent to the point one. But let's see what happens. Let's see if it actually does that.

So first let's apply F01 to it. That's going to pull this point up to this point here on M to the point one quarter comma one half. And then let's apply O2 to that point. That's going to push that point down along this line. And geometrically you can see that it does line up and go to one.

And you can verify that computationally if you want, algebraically. But it does seem to work. It goes there. So you can try this out on a couple more examples to convince yourself that it's true. It'll be useful if you work out two or three examples, you'll get a better sense of why this is taking every point to one point further.

Why it's translating points along this line. How many fixed points does this projectivity have? So as a hint, the translation function X goes to X plus one, that has no ordinary fixed points. Every point gets shifted along and nothing is fixed on the real line. Except maybe the point at infinity. If you add one to that point, you're still at infinity.

So we actually do have a fixed point if we include that point at infinity. So you can actually check for yourself that the point at infinity, if you imagine, just like we had earlier, you can see that the construction fixes it because both M and L meet at infinity. And the lines between through O1 and through O2, which are parallel, will hit that point at infinity. So that point at infinity is fixed by both F01 and F02. Now in general, a projectivity from an extended line to itself is called parabolic if there's exactly one fixed point.

So this is an example of a parabolic projectivity. So finally, let's look at a last third example. I want to try and reconstruct the function X goes to one over X through a projectivity. So I'm using a different line this time. I'm using the line connecting the point zero one with the point one zero.

So this is zero one. This is one zero. So I'm just reflecting those. It will also go through the point two minus one. And my centers of perspective are going to be the point O1, which is zero negative one over here, and O2, which is negative two comma one, negative two comma one, just sitting over here. So let's see how this projectivity acts on the real line. And let's do that by seeing how it acts on this point two comma zero.

So where does this point two comma zero get sent? By the perspective F01. So that's going to send it to some place on L, namely this point here. And where is this going to get sent? Through O2. Well, if we draw a line through O2 to here, we get to a point over here. I mean, sorry, we get to a point over here on the real line.

One half comma zero. So two is sent to one half comma zero. Two is sent to one half. So this seems to agree with the function x goes to one over x for that point two. And like in the other examples, try a couple more out. See where it sends one, see where it sends four, see where it sends three, and see if you can convince yourself that this construction will recreate the function x goes to one over x.

In particular, it's useful to see that it fixes one and negative one. This function fixes one and negative one. One over one is one. One over negative one is negative one.

So those are two fixed points. You can also check that it sends zero to infinity and that it sends infinity to zero. And since it has exactly two fixed points, this projectivity is also hyperbolic. It also has exactly two fixed points. Now, in general, a projectivity from a line to itself is called elliptic if it has zero fixed points, if it has no fixed points at all.

We haven't seen any example of that yet. And actually, they're a little harder to come by. They're a little harder to construct. So as a challenge, can you construct an elliptic projectivity on the line L? As a hint, you can try choosing any three points, A, B, and C. And you can construct a projectivity that permutes them cyclically. In other words, if this is A, this is B, and this is C, you can try and send A to B, B to C, and C to A.

And you have all the tools for doing that because we've seen that we know how to send any three points to any three other points through a projectivity. So you might have to add an extra line, another line or maybe even two lines, but you should be able to do that. And once you've done that, it's just a matter of proving that this map has no fixed points. So I want to make one last remark. So far, we've only looked at perspectives centered at ordinary points of R2.

But R2 is sitting inside of P2. And we have referred to those points at infinity lying outside of R2. But we can actually do more. We can actually consider perspectives whose centers are points at infinity. So the advantage of this is that we can construct perspectives that mimic parallel projection.

And that can simplify a lot of these constructions. Of course, we won't be able to do these constructions physically with a straight edge alone. We'll need to use a ruler. We'll need to use a measuring device like a compass, something that will let us draw parallel lines. But that's OK. If we're willing to do that, it does make some constructions easier.

So let's see an example. And let's see how this function x goes to 1 over x. This translation function can be represented by perspectives centered at points at infinity. So here's a point 0,1 at infinity. It's just a point at infinity corresponding to these 45 degree angle lines, that family of lines that has that angle. And where is that going to project? How is that going to act on the line L? If we think of a perspective centered at 0,1 from L to M, what is that going to look like? Well, it's going to send 0,0 up to 1,1.

It's going to send 1,0 up to 2,1 and so on. So it's going to match up points according to these 45 degree angles. And we could follow that with another perspective centered at infinity, f,0,2, where 0,2 is the point at infinity corresponding to vertical lines. That's just going to map things directly down from M to L. So 1,1 will get sent to 1,0.

2,1 will get sent to 2,0 and so on. So taken together, we're going to send 0,0 to 1,0. We're going to send 1,0 to 2,0 and so on. And any point x is going to get sent to x plus 1. That's a little easier to see than our previous construction of this function. So as an exercise, we've seen that the following functions can be constructed as projectivities.

x goes to 2x, x goes to x plus 1, and x goes to 1 over x. Can you construct arbitrary dilations and translations? Like x goes to ax, where a is any non-zero real number. Or x goes to x plus b, where b is any real number. And feel free to use points at infinity as centers of perspective, because that'll make some of these constructions easier. They can be done without points at infinity, but feel free to use them to simplify your construction.

So good luck with that. That's a useful exercise. And we'll use this result in our next video. Thank you.