Our Mathematical Senses

The Geometry Vision

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Lecture-41

Video 8E: the three fixed points theorem

Okay, so the last thing that I want to do is take a closer look at fixed points of projectivities. So we gave special names for projectivities that fix 0, 1, or 2 points. Elliptic means it fixes 0 points. Parabolic means it has a single fixed point. And hyperbolic means it has two fixed points. So what about projectivities that fix three or more points, like A, B, and C? What if they're all fixed by some projectivity from L back to itself? So according to a theorem that we're going to introduce now, the three fixed points theorem, a projectivity with three fixed points has a very specific form.

Namely, if a projectivity from L to itself fixes three distinct points, then it's actually the identity map on L. As an illustration, the sequence of perspectives from L1 back to itself, given by 0, 1, 0, 2, and 0, 3, I've just constructed here a sequence which fixes three points, A, B, and C. 0, 1 is going to send A to here, B to here, and C to here. So it's going to push A, B, and C to these three points in L2.

Then 0, 2 is going to pull these three points up into L3 to these three points here. Finally, 0, 3 is going to push these three points out back to L1, back to A, B, and C. So you can see that A, B, and C are fixed by this sequence of perspectives. And by the three fixed points theorem, that means that every single point in L1 must be fixed by this sequence of perspectives. If it fixes three points, it has to fix every point.

That's the content of this theorem. So how do we prove this? Well, there's actually several possible approaches. There's many proofs of the three fixed points theorem. We're going to sketch one proof today, and we're going to do a different proof fully in the next chapter. Or we're going to complete a different proof fully in the next chapter.

But that'll use a completely different approach. So we'll actually get two proofs of this. But today, we're going to do a proof based on linear fractional functions. And don't worry if you don't know what that means. I'm going to define it in a second.

But first, I want to just refresh our memory. So we've seen the following functions as projectivities. X goes to 2x, x goes to x plus 1, and x goes to 1 over x. We've seen how to construct those as projectivities. And as an exercise, I wanted you to check that arbitrary dilations and translations can also be constructed as projectivities.

In particular, x goes to ax, x goes to x plus b, and x goes to 1 over x. All of these functions can be constructed as projectivities according to that exercise. This is why it's a sketch of a proof. I'm leaving several holds for you to work out. But if we accept that, what do we get when we put these together? What other functions can we get as compositions of these three types of functions? Dilations, translations, and inversion.

Putting these together, we can construct any linear fractional function, where a linear fractional function is a function of the form x goes to ax plus b over cx plus d for real numbers a, b, c, and d, where there's one stipulation, ad minus bc is not equal to 0. The reason we need that is because if ad minus bc is equal to 0, then this function is going to be a constant function. I'll leave that as an exercise for you to check. If ad minus bc equals 0, this won't be very interesting. So for it to be a linear fractional function, this has to be non-zero.

So now let's sketch the proof. So doing some algebraic manipulation, you can work out that ax plus b over cx plus d can be rewritten as a over c plus bc minus ad over c times cx plus d. You can just take the right-hand side and manipulate it a bit to get to your left-hand side if you try and combine those fractions. It's not that hard to do, but I'll leave that as another exercise for you to check that this is actually the same as this. But the reason that this right-hand side formulation is more useful, is more illuminating, is that this is clearly a composition of scalar multiplication.

We're multiplying x by c, we're multiplying this by c, and scalar addition. We're adding cx to d, we are adding these two things, quantities. And finally, inversion. After getting this quantity, you'd have to take one over it, then multiply by.

.. So let me maybe, let me rewind a little bit, and you can see exactly how it's a composition by just following how the function's constructed. We take x, we multiply by c, we add d, we multiply all that by c, we do multiplicative inversion, we do one over that. We multiply that by bc minus ad, finally we add a plus c. So we can arrive at this expression through this composition. So in other words, we can get a function of this form as a projectivity.

But what's a little harder to see is that any projectivity will realize some linear fractional function. They're actually one and the same. Projectivities from the real line to itself are the linear fractional functions. So this is the sketchiest part of the proof. I'm leaving this as a whole for you to prove.

It's a challenge. To prove this by putting coordinates on two lines in R2 and working out a general equation for a perspectivity between them. So again, if you feel comfortable trying this, if you have the background to try this, go for it. But if you don't, don't worry. We can just accept this as a fact that projectivities from a line to itself are always linear fractional transformations if we write them algebraically. And by the way, the proof that we'll do in the, we'll do a completely different proof of the three fixed points theorem, which is fully based on stuff that we've done in this course.

So this is more just a kind of, just for your own illumination. So don't be distraught if this is suddenly involving things that you're not comfortable with. Just think of this as another way of seeing projectivities. And the reason this is helpful is that it's easy to understand the fixed points of a linear fractional function. A linear fractional function is just given by this expression.

If x is a fixed point, it means that whatever we put in for x, we get back from this expression. So x is actually equal to this expression. But if we expand, if we multiply this out and group everything on one side of the equation, we end up getting cx squared plus, let me just do that. We get that cx plus d times x is equal to ax plus b. Multiplying that out, we get cx squared plus dx minus ax minus b is equal to zero.

And then grouping these terms, we get exactly this expression. How many solutions are there for this equation though? Well, it's a quadratic equation, quadratic polynomial. So at most, there's two solutions. And most of there's two solutions for this, right? Well almost. It might happen that d is equal to a, so this disappears, and b and c are both zero.

Then we get zero equals zero and there's infinitely many solutions. But in that case, everything is a solution. Everything is a fixed point. And that simply is the identity map with infinitely many fixed points. And if that's not the case, then we have at most two solutions, which means there's at most two fixed points.

So there's either zero, one, or two fixed points, or infinitely many fixed points. There's four possibilities. So that does it. Now we're going to give a complete proof of this in the final week using a totally different strategy. But until then, we're going to assume this theorem holds.

And we'll see how powerful it is. It's going to lead to a lot of useful results. In particular, it'll allow us to prove Pappus' theorem and allow us to prove what's known as the fundamental theorem of projective geometry.