Our Mathematical Senses

The Geometry Vision

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Lecture-42

Video 9A: 1D fundamental theorem of projective geometry

In today's lecture, we're finally going to get to a theorem that we've mentioned several times already in the course, namely the fundamental theorem of projective geometry. But actually, it's going to come out of a theorem we just discussed in the end of the last lecture. Notice the three fixed points theorem. Now that stated that if a projectivity from a line L to itself fixes three distinct points, then it's actually the identity map on L. It fixes every point. And we sketched a proof of it in the last lecture.

And today we're not going to prove it yet, but rather we're going to see three extremely important applications which indicate just how fundamental this theorem, the three fixed points theorem is. And the first application that I want to discuss is the fundamental theorem of projective geometry, but the one-dimensional version, the version that relates to lines as opposed to planes. So given two lines, capital L and little l in P3, and given distinct points, a, b, and c on capital L, and little a, little b, little c on little l, there exists a unique projectivity from capital L to little l, which takes capital A to little a, capital B to little b, and capital C to little c. So I want to emphasize here that when I say projectivity, I'm referring to the map from L to L, not to the particular sequence of perspectives that were used to realize that map.

There may be many different sequences of perspectives that take a, b, and c to little a, little b, and little c. But in the end of the day, they're all going to agree on the final destination of x for any point capital X in L. Rather, let me say it again, the final destination little x in little l of any point capital X in L. So they'll agree as maps. We've already seen one construction of a projectivity that takes capital A, capital B, and capital C to little a, little b, and little c, where these points were chosen arbitrarily in capital L and little l.

The construction I'm talking about is this one, which mapped first using a perspective centered at O1, we sent a, b, and c to a, b prime and little c. And then using the second perspective, we pulled those into little a, little b, and little c. So we've seen one such projectivity, which takes three points on L to an arbitrary three points on little l. But so the existence part of this theorem we've already done. But to show the uniqueness part, to show that this is a unique projectivity, let's suppose that gamma is one projectivity from L to L that takes a, b, and c to a, b, and c.

And let's suppose that gamma prime is another projectivity taking a, b, and c to a, little a, little b, and little c. Then gamma prime followed by gamma inverse is a map from L to itself. And it's going to fix capital A, capital B, and capital C because gamma prime is going to send capital A to little a, and gamma inverse is then going to take little a back to capital A. And the same thing for b and c. They're each going to get sent back to themselves.

So this composition is going to fix a, b, and c. So by the three fixed points theorem, this composition map is the identity map. It's going to fix every single point on capital L. Now since gamma and gamma prime are both bijections, it follows that they have to be equal. Another way to think about it, gamma and gamma prime are two maps from capital L to little l.

Now, if we do gamma prime and then follow it by gamma inverse, we're getting the identity. Gamma inverse is literally undoing everything that gamma prime did. So the fact that it's gamma inverse, but gamma inverse is also the inverse of gamma. It also undoes everything gamma does. So as a result, gamma had to be equal to gamma prime in the first place.

So that takes care of the uniqueness part as well and proves the fundamental theorem of projective geometry in the 1D version from lines to lines.