Our Mathematical Senses

The Geometry Vision

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Lecture-47

Video 9F: the shadow of a square

So, let's return to our discussion from the intro video about the shadow of a square, the possible shadows that a square can cast. And we asked the question, what is the perspective image of a square? Well, the fundamental theorem gives a partial answer to that question. It tells us that the projective image of a square is actually any quadrilateral. Or more precisely, it tells us that an ordered set of four points in P3, as long as no three are collinear, can be taken to any other ordered set of four points in P3, provided no three points are collinear. So it gives us a very related, it's an answer to a very related question. But what if we actually want to know about the image of a square under a single perspective? Well, the answer to that actually also depends on whether we allow the center of perspective to be a point at infinity.

In other words, are we considering sunlight, or are we considering torchlight, or are we considering both? So it's a nice exercise to show that the shadow of a square under sunlight, in other words, the image of a square under a perspective whose center is infinitely far away, that can actually be any parallelogram. So if you actually tried to take a square outside and looked at its shadows under sunlight, you may have noticed that all of the shadows you're getting are very clearly parallelograms. And this type of projection is known as parallel projection, because the sun is so far away that it's effectively infinitely far away, and the light rays it's casting all appear to be parallel to each other. So it's a nice exercise to show that, in fact, the shadow of a square under parallel projection is any parallelogram.

You can get any parallelogram that way. On the other hand, another useful exercise is to show that the shadow of a square under torchlight, by torchlight I mean a single point source like the LED torchlight on your phone, so that this is actually any quadrilateral that's not a parallelogram. So it's kind of the other end of things. It's every other quadrilateral that's not a parallelogram. You can get this way through as a perspective or a shadow under torchlight.

And again, what I really mean is a perspective centered at an ordinary point, not a point at infinity, but an ordinary point. And this type of projection is known as central projection. So it's interesting, these are these two types of light sources with centers, these types of perspectives whose centers are infinitely far away or ordinary points, they give you almost disjoint sets of shadows. There is a small exception, which is that torchlight can give you one particular parallelogram, it can give you another square. If you take a square and center your torch right at the center of it, then and say project down onto a table, you will get a larger square.

So that's the one parallelogram you can get via central projection, which you can also get via parallel projection. But otherwise the shadows actually form disjoint sets of shapes. So both of these are nice exercises. They can be done from first principles. But if you want to see some details or some hints, there's a very nice paper which details this problem, which I'll print here.

It's by Cranel, France and Futamura. And it's just a paper called the image of a square. So in conclusion, we've seen that the proofs, both the proofs of the fundamental theorems and Pappus' theorem also, all three of these proofs relied on the three fixed points theorem. And we still haven't proved the three fixed points theorem. But we're going to do that in the next lecture.

And the way we're going to do it is to introduce a new numerical quantity known as the cross ratio, which is invariant under shifts in perspective. So see you then.