Our Mathematical Senses

The Geometry Vision

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Lecture-48

Chapter Four: The Analytic Framework

J QUARTcoT The word geometry literally means measuring the Earth, and yet until now our study of projective geometry has largely avoided any real use of numbers or measurements. Maybe we're just being lazy, or maybe we're actually being smart. After all, what use are numbers when lengths change completely from one perspective view to the next? OK, but what about relations between lengths, like ratios of lengths? Maybe those are preserved as we shift from one perspective to another. Well, that doesn't seem much better. Here we have two images of an evenly spaced square tiling. In the bird's eye view, all the tiles look identical, so the ratios of successive lengths are all equal to one.

But the ratios are clearly not equal to one in the second image. This brings up an interesting question, though. What is the relation between successive lengths in this image? It seems like a reasonable guess that the sequence of successive lengths might form a geometric progression. And yet, on closer examination, it doesn't.

I've measured out these tile lengths to be 30 units, 10 units, and 5 units. So the ratio between the first and second tile is 3, whereas the ratio between the second tile and the third tile is just 2. And measuring out the next length, we get 3 units. Definitely not a geometric progression. Here's a puzzle.

What is the subsequent tile length after 3 units? Can you find any pattern at all to these lengths? Pause the video if you'd like to take a moment to think about it. There is indeed a pattern to discover. Believe it or not, although the ratios of lengths are not preserved under perspective shifts, certain ratios of ratios are preserved. In particular, we'll soon learn how to calculate the cross ratio, which is sometimes referred to as the fundamental invariant of projective geometry. Almost magically, the cross ratio of four collinear points will always remain the same, no matter which perspective those points are viewed from.

We can define the cross ratio of four collinear points as follows. If the distances between the points are x, y, and z, then the cross ratio of the points is equal to x over y divided by x plus y plus z over negative z. Let's look at a simple example where x, y, and z are all equal, like this bird's eye view of the tiled floor. The cross ratio of these four collinear points comes out to be 1 divided by negative 3, or negative one-third. Now let's check the cross ratio of a different set of evenly spaced collinear points.

Measuring out distances in pixels, we get that segment x is 147 pixels long, segment y is 81 pixels long, and segment z is 51 pixels long. Computing the cross ratio with these values, we get 147 over 81 divided by 279 over negative 51, which miraculously comes out to negative one-third again. You can check it yourself. In fact, you can take any photograph of any four evenly spaced collinear points, and the cross ratio will always come out to negative one-third, no matter how strange the perspective is. This is because the cross ratio is invariant under perspective shifts, a property we will prove in the upcoming lecture, once we've defined the cross ratio a little more carefully.

The cross ratio was well known even to mathematicians of ancient Alexandria, like Pappus, Hypatia, and Pandrosian in the 4th century AD. And yet, until the 19th century, projective geometry was mainly studied without coordinates or equations, but rather via straight-edge drawings and axioms. That approach to geometry is sometimes called synthetic, because results are proved through explicit hands-on constructions. The alternate, analytic approach to geometry received a huge boost in popularity in 1637. When the mathematician Descartes published Le Geometry, giving birth to Cartesian coordinates and analytic geometry.

But projective geometry was not so easily framed in a coordinate system. I mean, how can we assign coordinates when distances are not even fixed? Instead, the synthetic approach, axioms, and straight-edge drawings, remain the primary mode of investigation. Until a remarkable new framework came into being at the end of the 19th century. The mathematician August Ferdinand Mobius introduced homogeneous coordinates, which allowed for a purely analytic study of projective geometry. Now, synthetic methods are primarily useful for mathematicians and artists, but using analytic methods, we can offer solutions to the wider world.

People like computer scientists, or engineers, or football players. The jus mesdames et sineurs have become quite the football fan. Ah, oui! I have become positively obsessed with the most legendary free kick in human history. I am talking about Brazil vs France 1997, when Roberto Carlos won the game for Brazil with a single kick. Sacre bleu! I was quite upset at first, but then I found myself appreciating the sheer majesty of that kick.

How on earth did he do it? Unfortunately, every report of his kick gives a different distance from the ball to the goal. Some say it is 34 meters, some say it is 37 meters, some say it is 40 meters. Which is the actual distance? It is driving me cock-hole! Can you please assist me using this photograph? Can we somehow calculate the actual distance from the ball to the goal? Oui? Dizag! Dizag! You could have asked me for help. You most certainly can calculate the distance using the cross ratio. We know that neither length nor even ratios of lengths are preserved and the changes in perspective, so those won't help us calculate the distance.

But the cross ratio is a ratio of ratios which miraculously is preserved. But now I too have a problem. I have become quite the avid photographer ever since I found the camera in your studio. I took a photo of one of my favourite theorems on the invariance of the cross ratio. But I was in a rush and the angle of my photo is completely off, making it rather painful to read.

There must be some way to fix it, to transform the image into the correct perspective. But how? How? I must admit I am at a loss. In fact, analytic methods will solve both these problems and many many more. In this final week of the Geometry of Vision, we'll explore the cross ratio as well as the analytic approach to projective geometry. To do so, we'll introduce the real projective plane, define homogeneous coordinates, and investigate the transformation group PGL3R. So let's begin.