

Our Mathematical Senses

The Geometry Vision

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Lecture-51

Video 10C: how many cross ratios?

Now, if you looked up the cross ratio on Wikipedia or in some textbook, you may have gotten a slightly different formula for it. And that's because there's actually multiple different ways of defining the cross ratio. So I want to make a remark that the order of the points matters in the cross ratio. Given four points, A, B, C, and D, we've defined the cross ratio, $AC \text{ ; } BD$, as $AB \text{ over } BC$, all divided by $AD \text{ over } DC$. That's our definition. Now, there's actually four factorial, or 24, different orderings of these points, A, B, C, and D.

And they don't all give the same cross ratio. What I mean is, for example, if we do the cross ratio $AC \text{ ; } DB$, if we interchange D and B effectively in our formula, what do we get? Well, let me just write out the cross ratio formula a little more clearly here. So it's $AB \text{ over } BC \text{ times } DC \text{ over } AD$. I flipped this fraction so that I can write it as a multiplication here.

So if we interchange B and D, as I'm doing here, what happens? If we interchange D and B, we end up with $AD \text{ over } DC \text{ times } BC \text{ over } AB$. How are these two quantities related? Well, you can see they're actually reciprocals of each other. Over here, the top is $AB \text{ DC}$, $AB \text{ times } DC$. Here, the bottom is $DC \text{ times } AB$. Over here, the bottom is $BC \text{ times } AD$.

Over here, the top is $AD \text{ times } BC$. So these are reciprocals. So this cross ratio, if I take the points in this order, will be the reciprocal of what I would have gotten if I took the points in this order. So the order matters. I'll get different numerical outputs if I input the points in a different order.

Similarly, if we were to use this order and this ABCD, I'll leave it to you to calculate this. But effectively, what we're doing here is interchanging the points B and C from our original formula. So if you take that, interchange the points B and C, you end up with 1 minus our familiar cross ratio. So for four points, our original definition gave us a cross ratio of negative 1 third, if they're evenly spaced, for example. Using this cross ratio, we'll get 1 minus negative 1 third, which is 4 third, positive 4 thirds.

So that's what this cross ratio will give us for a set of four evenly spaced points. And incidentally, this is Wikipedia's chosen cross ratio. And I will admit, this is a slightly more common formulation of the cross ratio. I would say the one that we're using is maybe the second most common. But it's still, it's, yeah, each of them has advantages and disadvantages, is all I'll say at the moment.

But if you look at Wikipedia, don't be confused if you see this cross ratio. It's very, very related to ours. And this is how they're related. On the other hand, some permutations of ABCD won't change the numerical output that you get. All of these will give you the same cross ratio that we defined here, all of these permutations.

So all in all, there's 4 factorial, or 24 different ways to define it. But really, there's only six different possible values we could have ended up with, because many of these will give the same value as each other. So if ACBD is λ , then the other possible cross ratio values you could get by permuting the points are $1 - \lambda$, $1/\lambda$, or the things you get by combining those, $1/(1 - \lambda)$, $\lambda/(1 - \lambda)$, or $\lambda/(\lambda - 1)$. So there's six different cross ratios, and all of those are indeed invariants. But they're all highly related to each other.