**Our Mathematical Senses**

**The Geometry Vision**

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**Lecture-52**

Video 10D: invariance of the cross ratio

So, let's turn our attention now to the invariance of the cross ratio. This is the theorem or the property that tells us that the cross ratio is preserved under changes in perspective. Another way of saying that, prospectivities preserve the cross ratio. So, in order to do this, we want to use a dual notion of the cross ratio that I'm going to introduce now. So up till now we've been working with a cross ratio of points. I want to introduce a cross ratio of lines.

So in particular, if we have four concurrent coplanar lines, A, B, C, and D, I want to define a cross ratio of lines in the following way. So this is going to be based on signs of angles. So I want to look at the angles APB, BPC, APD, and DPC, where the last angle is going to have a negative value because it's a signed angle. So just analogous to how we had signed lengths earlier, we're also going to have signed angles.

And this is going to be negative because it's going in the wrong direction. So the cross ratio of lines is going to be defined as follows. It's the sine of APB divided by the sine of BPC all over the sine of APD divided by the sine of the angle DPC of that negative angle. So that's the cross ratio of lengths. Now the cross ratio duality lemma is going to tell us the following.

That if we take four concurrent collinear lines and we intersect them with another line here, and if we call the points of intersection A, B, C, and D, then the cross ratio of the points A, B, C, and D is going to be equal to the cross ratio of the lines A, B, C, and D. So in fact, these two notions of cross ratio coincide. So that's what we want to prove now. And I'm just going to keep the definition of these two cross ratios here so it's handy. It's going to be very, it's a complicated definition.

It's easy to forget. So let's just keep it up here. And let's see how to prove, I'm going to hide the actual statement of the lemma. But the lemma is just saying that these two cross ratios are in fact equal. Now the proof is from the website CutTheKnot.

You can find it by searching for the cross ratio on CutTheKnot. And the proof itself, it's transmitted by Alexander Bogomolny, but I believe it might be much older. It might even be classical in its origin. So the method of the proof is to introduce a third version of the cross ratio based on area. So let's consider four triangles, APB, BPC, APD, and DPC.

And let's represent the areas of these triangles in two different ways. So we want to look at the area of APB, the area of APD, BPC, APD, and DPC. And this last one is going to be a negative because it's, again, in the wrong direction. So one way of representing the areas is just the base times the height over two. The area of APB is one half times the base times the height, which I've marked here.

Similarly, this triangle BPC, its area is going to be BC times H over two. The area of APD is just going to be AD times H over two. And the area of DPC is going to be DC, which is negative, times H over two. So this is one way of representing the lengths of the areas of the triangles. There's another way of representing the areas of the triangles, though, which is given here.

You might have seen this in a high school trigonometry class, for example. We can also write the area of APB as PA times PB times the sine of APB over two. And similarly, we can write the area of BPC as PB times PC times the sine of BPC over two. And so on. The area of the big triangle, it's PA times PD times the sine of APD over two.

And finally, this triangle has area PD times PC times the sine of this negative angle DPC over two. So we have another way of representing all four of these areas. Now the interesting thing is that if we group these, this looks quite complicated and unhelpful at the moment, but let's group these expressions in a way that will be a little bit more enlightening. So let's consider the ratio of these top two guys, and let's divide that by the ratio of these bottom two expressions here. And when we do that, what do we see? Well, let's focus our attention on this middle set of terms.

Let's ignore everything else and look at these middle expressions. So we're looking at the ratio of this to this over the ratio of this to this. Well, some things cancel out. Maybe you see that these H's are just going to cancel right out. And similarly, these one halves are going to cancel right out.

And we end up with AB over BC divided by AD over DC. Does that look familiar at all? It's AB over BC divided by AD over DC. It's exactly our cross ratio of points. So our cross ratio of points is this central set of expressions right here, this central ratio. Now what about our cross ratio of last, what about this third set of expressions? So let's just take a closer look at that.

Well, again, many things seem to cancel out. The one halves all cancel out, so we can ignore those. So if we look at just this top ratio, let's just focus on this top ratio here. We have a PB on the top and a PB on the bottom. Those are going to cancel out.

Similarly, on the bottom expression, we have a PD on the top and a PD on the bottom. Those are going to cancel out. Now in the top fraction, we have a PA over PC. In the bottom fraction, we also have a PA over PC. So those are going to cancel out.

PA over PC, PA over PC. And we're left with sine of APB divided by sine of BPC all over sine of APD divided by sine of DPC. We're left with the cross ratio of lines. That expression is equal to both the cross ratio of points and the cross ratio of lines, showing that those are both equal to each other. So it follows that there is really just one cross ratio. And the invariance of the cross ratio follows quite nicely from there.

So the theorem now is that the cross ratio of collinear points does not change under a perspective. And how do we see this? Well, let's just take four points, A, B, C, and D. Let's just imagine a perspective centered at P between points A, B, C, and D, and another set of points, A prime, B prime, C prime, and D prime. That perspective is mapping A prime and A, B prime to B, C prime to C, and D prime to D. In other words, there's lines from that center of perspective.

There's a line A that connects A prime and A, a line B, script B, which connects B prime and B, a line script C, which connects C prime and C, and a line script D, which connects the points D prime to D. But we've just seen from the proof, we had almost the same diagram. The cross ratio of the points A, B, C, and D is equal to the cross ratio of the lines, script A, script B, script C, script D. And that, in turn, is equal to the cross ratio of the points A prime, B prime, C prime, and D prime. So the cross ratio of these four points is equal to the cross ratio of these four points.

In particular, remember, in the cross ratio duality lemma, this line here was just any line that intersected these four lines. It could have been this line, it could have been this line, it could have been this. There's nothing special about this line that we used. So this line and this line both give us points whose cross ratios are equal to the cross ratio of lines. And that proves that the cross ratio is invariant under perspectives.